The University of Texas at Austin
In-Class Work 15

M378K Introduction to Mathematical Statistics

March 28, 2026

Bias. MSE.

Problem 15.1. *Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.* Let $Y_1,, Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

Solution. First, let us check for unbiasedness.

$$\mathbb{E}\left[\hat{\theta}_1\right] = \mathbb{E}[Y_1] = \tau$$
$$\mathbb{E}\left[\hat{\theta}_2\right] = \mathbb{E}\left[\frac{Y_1 + Y_2}{2}\right] = \tau$$
$$\mathbb{E}\left[\hat{\theta}_3\right] = \mathbb{E}\left[\frac{Y_1 + 2Y_2}{3}\right] = \tau$$
$$\mathbb{E}\left[\hat{\theta}_4\right] = \mathbb{E}\left[3Y_{(1)}\right] = 3 \cdot \frac{\tau}{3} = \tau$$
$$\mathbb{E}\left[\hat{\theta}_5\right] = \mathbb{E}\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \tau$$

So, all of the offered estimators are unbiased.

Now, for the variance, we have

$$\text{Var}\left[\hat{\theta}_1\right] = \text{Var}\left[Y_1\right] = \tau^2$$
$$\text{Var}\left[\hat{\theta}_2\right] = \text{Var}\left[\frac{Y_1 + Y_2}{2}\right] = \frac{2\tau^2}{4} = \frac{\tau^2}{2}$$
$$\text{Var}\left[\hat{\theta}_3\right] = \text{Var}\left[\frac{Y_1 + 2Y_2}{3}\right] = \frac{1}{9}(\tau^2 + 4\tau^2) = \frac{5\tau^2}{9}$$
$$\text{Var}\left[\hat{\theta}_4\right] = \text{Var}\left[3Y_{(1)}\right] = 9 \cdot \left(\frac{\tau}{3}\right)^2 = \tau^2$$
$$\text{Var}\left[\hat{\theta}_5\right] = \text{Var}\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \frac{\tau^2}{3}$$

The sample mean has the minimum variance.

Problem 15.2. Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased for the same parameter θ. We know that $\text{Var}\left[\hat{\theta}_1\right] = \sigma_1^2$ and $\text{Var}\left[\hat{\theta}_2\right] = \sigma_2^2$.

Consider all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the *bias* of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are **independent**, for which weight α is the variance minimal?

Solution. Due to the linearity of expectation, we have

$$\mathbb{E}\left[\hat{\theta}\right] = \alpha\mathbb{E}\left[\hat{\theta}_1\right] + (1-\alpha)\mathbb{E}\left[\hat{\theta}_2\right] = \theta.$$

So, each estimator of the above form is *unbiased*. Now, let's consider the variance. We want to solve this optimization problem

$$\begin{gathered}\text{Var}\left[\hat{\theta}\right] \to \min \\ \Leftrightarrow \\ \text{Var}\left[\alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2\right] \to \min\end{gathered}$$

Due to f independence of $\hat{\theta}_1$ and $\hat{\theta}_2$, the above is equivalent to

$$\begin{gathered}\alpha^2\ \text{Var}\left[\hat{\theta}_1\right] + (1-\alpha)^2\ \text{Var}\left[\hat{\theta}_2\right] \to \min \Leftrightarrow \\ \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 \to \min\end{gathered}$$

This is an upward facing parabola in α. So, if we differentiate with respect to α, set our derivative equal to zero, and solve for α we will figure out the minimum.

$$2\alpha\sigma_1^2 - 2(1-\alpha)\sigma_2^2 = 0 \quad\Leftrightarrow\quad \alpha(\sigma_1^2+\sigma_2^2) = \sigma_2^2 \quad\Leftrightarrow\quad \alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}.$$

Problem 15.3. Let $Y_1, Y_2, ..., Y_n$ be a random sample from a continuous distribution with probability density function

$$f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} 1_{()[0,\theta]}(y)$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, ..., Y_n)$. Is this estimator *unbiased*? If not, how would you modify it to create an *unbiased* estimator? What is the **mean-squared error** of the *unbiased* estimator you obtained?

Solution. The proposed estimator is the n^{th} order statistic. So, let's start by figuring out its density; it will be, for $y \in [0, \theta]$,

$$g_{(n)}(y) = n f_Y(y)(F_Y(y))^{n-1}$$

where F_Y stands for the cumulative distribution function of Y. In fact, for $y \in [0, \theta]$, we have that

$$F_Y(y) = \int_0^y f_Y(u)\,\mathrm{d}u = \frac{\alpha}{\theta^\alpha}\int_0^y u^{\alpha-1}\,\mathrm{d}u = \frac{\alpha}{\theta^\alpha}\cdot\frac{y^\alpha}{\alpha} = \left(\frac{y}{\theta}\right)^\alpha$$

This is the reason that distributions from this family are called f power distributions. Now, we have that, for $y \in [0, \theta]$,

$$g_{(n)}(y) = n\frac{\alpha y^{\alpha-1}}{\theta^{\alpha}}\left(\frac{y}{\theta}\right)^{\alpha(n-1)} = \frac{n\alpha}{\theta^{\alpha n}}y^{\alpha n-1}$$

Hence,

$$\mathbb{E}\left[\hat{\theta}\right] = \int_0^{\theta} y g_{(n)}(y)\,\mathrm{d}y = \int_0^{\theta} y\frac{n\alpha}{\theta^{\alpha n}}y^{\alpha n-1}\,\mathrm{d}y = \frac{n\alpha}{\theta^{\alpha n}}\int_0^{\theta} y^{\alpha n}\,\mathrm{d}y = \frac{n\alpha}{\theta^{\alpha n}}\cdot\frac{\theta^{\alpha n+1}}{\alpha n+1} = \frac{\alpha n}{\alpha n+1}\theta \neq \theta.$$

So, our estimator is *biased*, but if we instead consider

$$\hat{\theta}' = \frac{\alpha n+1}{\alpha n}\hat{\theta},$$

this estimator f will be *unbiased*. By the proposition from class, we know that

$$MSE\left[\hat{\theta}'\right] = \mathrm{Var}\left(\hat{\theta}'\right) = \left(\frac{\alpha n+1}{\alpha n}\right)^2 \mathrm{Var}\left[\hat{\theta}\right].$$

Focusing on the second moment of $\hat{\theta}$, we obtain

$$\mathbb{E}\left[\hat{\theta}^2\right] = \int_0^{\theta} y^2 g_{(n)}(y)\,\mathrm{d}y = \int_0^{\theta} y^2\frac{n\alpha}{\theta^{\alpha n}}y^{\alpha n-1}\,\mathrm{d}y = \frac{n\alpha}{\theta^{\alpha n}}\int_0^{\theta} y^{\alpha n+1}\,\mathrm{d}y = \frac{n\alpha}{\theta^{\alpha n}}\frac{\theta^{\alpha n+2}}{\alpha n+2} = \frac{\alpha n}{\alpha n+2}\theta^2.$$

Finally,

$$\mathrm{Var}\left[\hat{\theta}\right] = \frac{\alpha n}{\alpha n+2}\theta^2 - \left(\frac{\alpha n}{\alpha n+1}\right)^2\theta^2$$

and

$$\mathrm{Var}\left[\hat{\theta}'\right] = \left(\frac{\alpha n+1}{\alpha n}\right)^2\left(\frac{\alpha n}{\alpha n+2}\theta^2 - \left(\frac{\alpha n}{\alpha n+1}\right)^2\theta^2\right) = \left(\frac{(\alpha n+1)^2}{\alpha n(\alpha n+2)} - 1\right)\theta^2 = \frac{1}{\alpha n(\alpha n+2)}\theta^2.$$