M378K Introduction to Mathematical Statistics
Problem Set #4
Expectation and variance: the discrete case.

Problem 4.1. Source: Sample P exam, Problem #481.
The number of days required for a damage control team to locate and repair a leak in the hull of a ship is modeled by a discrete random variable N. N is uniformly distributed on $\{1, 2, 3, 4, 5\}$.

The cost of locating and repairing a leak is $N^2 + N + 1$.

Calculate the expected cost of locating and repairing a leak in the hull of the ship.

$$E[N] = \frac{1}{5}(1+2+3+4+5) = 3$$

$$E[N^2+N+1] = E[N^2] + E[N] + E[1]$$

$$E[N^2] = \frac{1}{5}(1^2+2^2+3^2+4^2+5^2) = 11$$

$$E[N^2+N+1] = 11 + 3 + 1 = \boxed{15}$$

□

Review:

$$\boxed{Var[Y] = \mathbb{E}[(Y - \mathbb{E}[Y])^2] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2}$$

Example. Let Y have a finite variance.
Then, $Var[Y+Y] = Var[2Y] = 2^2 Var[Y]$

Theorem. Let Y be a r.v. w/ a finite variance, and let α be a real constant.
Then,
$$\boxed{Var[\alpha \cdot Y] = \alpha^2 \cdot Var[Y]}$$

Def'n. If two r.v.s Y_1 and Y_2 satisfy that
$$\mathbb{P}[Y_1 \in B_1, Y_2 \in B_2] = \mathbb{P}[Y_1 \in B_1] \cdot \mathbb{P}[Y_2 \in B_2]$$
for "all" $B_1, B_2 \subseteq \mathbb{R}$,
then, we say that Y_1 and Y_2 are independent.

Theorem. If Y_1 and Y_2 are independent, then,
$$\boxed{Var[Y_1 + Y_2] = Var[Y_1] + Var[Y_2]}$$

Example. Binomial

$Y \sim b(n,p)$

$Var[Y] = ?$

Idea #1. $\mathbb{E}[Y^2] = \sum_{k=0}^{n} k \cdot p_Y(k) = \sum_{k=0}^{n} \left(k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}\right)$

Idea #2. $I_j, j = 1..n$ are independent and $I_j \sim B(p)$

$Y = I_1 + I_2 + \cdots + I_n$

$Var[Y] = Var[I_1] + \dots + Var[I_n] = n \cdot p(1-p)$

$$\boxed{Var[Y] = n \cdot p \cdot (1-p)}$$

Problem 4.2. Source: Sample P exam, Problem #458.

An investor wants to purchase a total of ten units of two assets, A and B, with annual payoffs **per unit** *purchased of X and Y, respectively. Each asset has the same purchase price per unit. The payoffs are independent random variables with equal expected values and with* $\mathrm{Var}(X) = 30$ *and* $\mathrm{Var}(Y) = 20$.

Calculate the number of units of asset A the investor should purchase to minimize the variance of the total payoff.

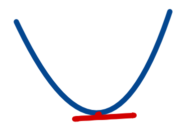
$\rightarrow$: n ... # of units of A that's bought

$10-n$ ... # of units of B bought

$$\mathrm{Var}[n\cdot X + (10-n)\cdot Y] \xrightarrow[n]{} \min$$

independence

$$n^2 \cdot \mathrm{Var}[X] + (10-n)^2 \cdot \mathrm{Var}[Y] \xrightarrow[n]{} \min$$



$$30n^2 + 20(10-n)^2 \xrightarrow[n]{} \min$$

$$30\cdot 2n + 20\cdot 2(-1)(10-n) = 0 \quad /:20$$

$$3n - 2(10-n) = 0$$

$$3n + 2n = 20$$

$$\boxed{n=4}$$

□

Example. • Geometric. $\boxed{Y \sim g(p)}$

$$\mathbb{E}[Y] = \frac{q}{p}$$

$$SD[Y] = \frac{\sqrt{q}}{p}$$

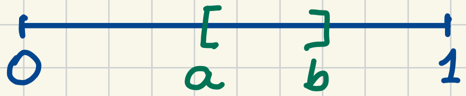
• Poisson. $\boxed{Y \sim P(\lambda)}$

$$\mathbb{E}[Y] = \mathrm{Var}[Y] = np(1-p)$$

Binomial $n\cdot p \approx np(\underbrace{1-\underset{\downarrow 0}{p}}_{1})$ rare events

Continuous Distributions.

The Uniform Distribution.



Imagine a r.v. Y on $[0,1]$ such that the probability of Y landing between a and b where $0 \le a \le b \le 1$ is

$$\boxed{\mathbb{P}[a \le Y \le b] = \mathbb{P}[Y \in [a,b]] = b-a}$$

Note: $\mathbb{P}[Y=y] = \mathbb{P}[y \le Y \le y] = y - y = 0$ for all $y \in [0,1]$.