October 28th, 2026.

Problem 15.3 [cont'd].

Focus on $\hat{\theta}' = \frac{n\alpha+1}{n\alpha} \cdot \hat{\theta} = \frac{n\alpha+1}{n\alpha} \cdot Y_{(n)}$

$MSE[\hat{\theta}'] = ?$

$MSE[\hat{\theta}'] = \boxed{Var[\hat{\theta}']} + \underbrace{(bias(\hat{\theta}'))^2}_{=0}$

$Var[\hat{\theta}'] = Var\left[\frac{n\alpha+1}{n\alpha} \cdot Y_{(n)}\right] = \left(\frac{n\alpha+1}{n\alpha}\right)^2 \boxed{Var[Y_{(n)}]}$

$Var[Y_{(n)}] = \boxed{E[Y_{(n)}^2]} - \underbrace{(E[Y_{(n)}])^2}_{\checkmark}$

$\int_0^\theta y^{n\alpha+1}\,dy = \frac{y^{n\alpha+2}}{n\alpha+2}\Big|_0^\theta$

$$\int_0^\theta y^2 g_{(n)}(y)\,dy = \int_0^\theta y^2 \cdot n \cdot \left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} \cdot \frac{\alpha \cdot y^{\alpha-1}}{\theta^\alpha}\,dy$$

$$= \frac{n\alpha}{\theta^{n\alpha}} \int_0^\theta y^{n\alpha+1}\,dy = \frac{n\alpha}{\theta^{n\alpha}} \cdot \frac{\theta^{n\alpha+2}}{n\alpha+2} = \frac{n\alpha}{n\alpha+2} \cdot \theta^2$$

$$Var[Y_{(n)}] = \frac{n\alpha}{n\alpha+2}\theta^2 - \left(\frac{n\alpha}{n\alpha+1}\right)^2 \cdot \theta^2$$

$$MSE[\hat{\theta}'] = \boxed{\left(\frac{n\alpha+1}{n\alpha}\right)^2} \left(\frac{n\alpha}{n\alpha+2} - \left(\frac{n\alpha}{n\alpha+1}\right)^2\right) \cdot \theta^2$$

$$= \left(\frac{(n\alpha+1)^2}{n\alpha(n\alpha+2)} - 1\right)\theta^2$$

$$= \frac{(n\alpha+1)^2 - n\alpha(n\alpha+2)}{n\alpha(n\alpha+2)} \cdot \theta^2$$

$$= \frac{\cancel{(n\alpha)^2} + \cancel{2n\alpha} + 1 - \cancel{(n\alpha)^2} - \cancel{2n\alpha}}{n\alpha(n\alpha+2)} \cdot \Theta^2$$

$$= \frac{1}{n\alpha(n\alpha+2)} \cdot \Theta^2$$

□

Point vs. Interval Estimators

Example 10.1.1. Consider the following two data sets, both consisting of measurements of the same quantity (say the distance to Proxima Centauri) and in the same units, but made with two different methods.

method 1: 4.51, 4.52, 4.48, 4.49, 4.47, 4.53
method 2: 14.12, 1.30, 0.40, 2.50, 1.00, 3.18

If we use the sample mean $\bar{Y}$ as the (point) estimator for the "true" mean μ, both of these data sets yield the same result, namely $\bar{Y} = 4.5$. It is clear, however, that the first method is more accurate and that, in general, one should trust the results produced by method 1 more than those obtained by method 2.

An interval estimator is a pair of point estimators such that $\hat{\theta}_L \leq \hat{\theta}_R$.

Desirable traits:

- "contains" the true parameter θ "w/ a high probability"
- is "narrow"

First: Pick a confidence level $C = 0.95, 0.99$, or another probability "close to 1",
or pick a significance level $\alpha = 0.05, 0.01$, or another probability "close to 0",

Convention: $C = 1 - \alpha$

The purpose:

$$\mathbb{P}\left[\hat{\theta}_L \leq \theta \leq \hat{\theta}_R\right] = C = 1 - \alpha$$

Def'n. Consider a random sample $(Y_1, \dots, Y_n)$ from a dist'n $\mathcal{D}$ which is parameterized by an unknown parameter θ.

A PIVOTAL QUANTITY is a function of $(Y_1, \dots, Y_n)$ and the parameter θ whose distribution does not depend on θ.

Example. Say that $Y_i \sim N(\mu, 1)$, $i = 1..n$, is our random sample.

$\bar{Y}$ is a great estimator for μ

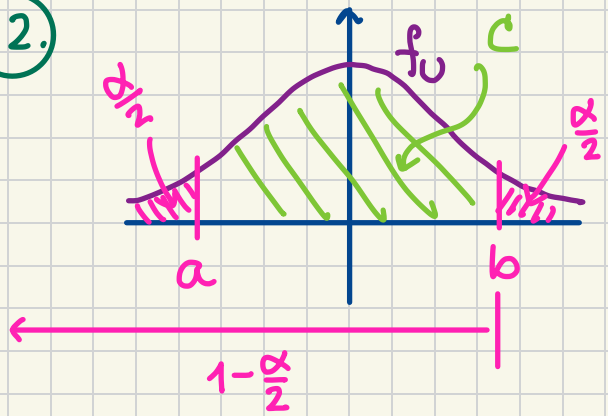
$\bar{Y} \sim \text{Normal}\left(\mu, \text{sd} = \frac{1}{\sqrt{n}}\right) \Rightarrow \bar{Y}$ is NOT A PIVOTAL QUANTITY

BUT $\boxed{\bar{Y} - \mu \sim \text{Normal}\left(0, \frac{1}{\sqrt{n}}\right)}$ is a PIVOTAL QUANTITY.

Example. $Y_1, \dots, Y_n \sim N(\mu, \sigma)$ w/ both parameters unknown

$$\boxed{\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)}$$ is a PIVOTAL QUANTITY for $\theta = (\mu, \sigma)$

Recipe:

1. Consider a pivotal quantity U
2. 

Define $a = q_U(\alpha/2)$ and $b = q_U(1-\frac{\alpha}{2})$

w/ q_U the quantile f'tion of U

Now, we know: $\mathbb{P}[a \le U \le b] = C = 1-\alpha$

Has θ in its expression!

3. ALGEBRA "Undo" the "rule" for U so that θ is isolated and your final expression looks like

$$\mathbb{P}\left[\hat{\theta}_L \leq \theta \leq \hat{\theta}_R\right] = C = 1-\alpha$$

Works for 1-D Parameters!