$$\text{Var}[\hat{\theta}_1] = \text{Var}[\underbrace{Y_1}_{\sim E(\tau)}] = \tau^2$$

$$\text{Var}[\hat{\theta}_2] = \text{Var}\left[\frac{Y_1+Y_2}{2}\right] = \frac{1}{4}\text{Var}[Y_1+Y_2] \overset{\text{independent}}{=} \frac{1}{4}(\text{Var}[Y_1]+\text{Var}[Y_2]) = \frac{\tau^2}{2}$$

$$\text{Var}[\hat{\theta}_3] = \text{Var}\left[\frac{Y_1+2Y_2}{3}\right] = \frac{1}{9}(\underbrace{\text{Var}[Y_1]}_{\tau^2} + 4\cdot\underbrace{\text{Var}[Y_2]}_{\tau^2}) = \frac{5}{9}\tau^2$$

$$\text{Var}[\hat{\theta}_4] = \text{Var}[3\cdot Y_{(1)}] = 9\cdot\text{Var}[\underbrace{Y_{(1)}}_{\sim E(\frac{\tau}{n})}] = 9\cdot\frac{\tau^2}{9} = \tau^2 \quad (n=3)$$

$$\text{Var}[\hat{\theta}_5] = \text{Var}[\bar{Y}] = \frac{1}{9}\cdot(3\tau^2) = \boxed{\frac{\tau^2}{3}}$$

□

Remark: When we want to estimate the mean,

$$\mathbb{E}[\bar{Y}] = \text{population mean}, \text{ i.e., } \underline{\text{unbiased}},$$

$$\text{MSE}[\bar{Y}] = \text{Var}[\bar{Y}] = \boxed{\frac{\text{Var}[Y_1]}{n}}$$

$$\Rightarrow \boxed{\text{SE}[\bar{Y}] = \frac{\text{SD}[Y_1]}{\sqrt{n}}}$$

Problem 15.2. *Suppose that the two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased. We know that $\mathrm{Var}[\hat{\theta}_1] = \sigma_1^2$ and $\mathrm{Var}[\hat{\theta}_2] = \sigma_2^2$.*

Consider ~~the estimator~~ all the estimators that can be obtained as convex combinations of $\hat{\theta}_1$ and $\hat{\theta}_2$, i.e., all the estimators of the form

$$\hat{\theta} = \alpha\hat{\theta}_1 + (1-\alpha)\hat{\theta}_2.$$

What can you say about the bias of estimators $\hat{\theta}$ of the form above? Assuming that $\hat{\theta}_1$ and $\hat{\theta}_2$ are **independent** *for which weight α is the variance minimal?*

$\rightarrow$: $\mathbb{E}[\hat{\Theta}] = \mathbb{E}[\alpha\hat{\Theta}_1 + (1-\alpha)\hat{\Theta}_2] = \alpha\,\mathbb{E}[\hat{\Theta}_1] + (1-\alpha)\,\mathbb{E}[\hat{\Theta}_2] = \theta$

(θ over $\mathbb{E}[\hat{\Theta}_1]$, θ over $\mathbb{E}[\hat{\Theta}_2]$)

linearity of $\mathbb{E}$

$\Rightarrow \hat{\Theta}$ is unbiased for all α

$$\mathrm{Var}[\hat{\Theta}] \xrightarrow[\alpha]{} \min$$

$$\mathrm{Var}[\alpha\hat{\Theta}_1 + (1-\alpha)\hat{\Theta}_2] \xrightarrow[\alpha]{} \min$$

independence

$$\alpha^2 \cdot \mathrm{Var}[\hat{\Theta}_1] + (1-\alpha)^2 \cdot \mathrm{Var}[\hat{\Theta}_2] \xrightarrow[\alpha]{} \min$$

$$\alpha^2 \cdot \sigma_1^2 + (1-\alpha)^2 \cdot \sigma_2^2 \xrightarrow[\alpha]{} \min$$

$$2\alpha\sigma_1^2 + 2(1-\alpha)(-1)\cdot\sigma_2^2 = 0$$

$$\alpha(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$\boxed{\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

□

Def'n. An estimator $\hat{\Theta}$ of Θ is said to be **linear** if it's of the form

$$\hat{\Theta} = \alpha_1 Y_1 + \alpha_2 Y_2 + \cdots + \alpha_n Y_n$$

where α_i, $i = 1..n$, are all constants.

e.g.; $\bar{Y}$;

Counterexample: sample variance, $x \cdot Y_{(1)}$

Def'n. An estimator $\hat{\Theta}$ of Θ is said to be

uniformly minimum variance unbiased estimator (UMVUE)

if
- $\hat{\Theta}$ is unbiased

and
- $MSE(\hat{\Theta}) \leq MSE(\hat{\Theta}')$ for all other unbiased estimators $\hat{\Theta}'$ of Θ

Example. These are UMVUE:

- $\bar{Y}$ for μ where $(Y_1, \ldots, Y_n)$ is a random sample from $N(\mu, \sigma)$
- $\bar{Y}$ for p where $(Y_1, \ldots, Y_n)$ is a random sample from $B(p)$.
- $$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

  for σ^2 where $(Y_1, \ldots, Y_n)$ is a random sample from $N(\mu, \sigma)$ w/ both parameters unknown

Problem 15.3. *Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a continuous distribution with probability density function*

$$f_Y(y) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \mathbf{1}_{[0,\theta]}(y)$$

with a known parameter $\alpha > 0$ and an unknown parameter $\theta > 0$. We propose the estimator $\hat{\theta} = \max(Y_1, \ldots, Y_n)$. Is this estimator unbiased*? If not, how would you modify it to create an* unbiased *estimator? What is the* **mean-squared error** *of the* unbiased *estimator you obtained?*

$\hat{\Theta} = \max(Y_1, \ldots, Y_n)$

$\mathbb{E}[\hat{\Theta}] = \mathbb{E}[Y_{(n)}] = ?$

$g_{(n)} = n \cdot (F_Y(y))^{n-1} \cdot f_Y(y)$ for $y \in (0, \theta)$

$F_Y(y) = \int_0^y f_Y(u)\,du$

$= \int_0^y \frac{\alpha}{\theta^\alpha} u^{\alpha-1}\,du$

$= \frac{\alpha}{\theta^\alpha} \cdot \left(\frac{u^\alpha}{\alpha}\right)_{u=0}^{y}$

$= \left(\frac{y}{\theta}\right)^\alpha$

$\mathbb{E}[Y_{(n)}] = \int_0^\theta y \cdot g_{(n)}(y)\,dy$

$= \int_0^\theta y \cdot n \left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} \cdot \frac{\alpha \cdot y^{\alpha-1}}{\theta^\alpha}\,dy$

$= n \cdot \alpha \cdot \frac{1}{\theta^{\alpha n}} \int_0^\theta y^{n\alpha}\,dy = n \cdot \alpha \cdot \frac{1}{\theta^{\alpha n}} \cdot \frac{\theta^{n\alpha+1}}{n\alpha+1}$

$\mathbb{E}[Y_{(n)}] = \frac{n\alpha}{n\alpha+1} \cdot \theta$

Not unbiased!

Let $\hat{\Theta}'$ be the unbiased "modification" of $\hat{\Theta}$.

$$\hat{\Theta}' = \frac{n\alpha+1}{n\alpha} \cdot \Theta$$