Bias [Cont'd].

Example. Let $Y_1, \dots, Y_n$ be a random sample from $N(\mu, \sigma^2)$ w/ both parameters unknown.

Goal: Find a "good" estimator for σ^2!

Propose:
$$(S')^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Q: Is $(S')^2$ unbiased for σ^2?

$$\mathbb{E}[(S')^2] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(Y_i-\bar{Y})^2]$$

$$\mathbb{E}[Y_i^2 - 2Y_i\cdot\bar{Y} + \bar{Y}^2] = \mathbb{E}[Y_i^2] - 2\mathbb{E}[Y_i\cdot\bar{Y}] + \mathbb{E}[\bar{Y}^2]$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbb{E}[Y_i^2] - 2\sum_{i=1}^{n}\mathbb{E}[Y_i\cdot\bar{Y}] + \sum_{i=1}^{n}\mathbb{E}[\bar{Y}^2]\right), \quad \sum_{i=1}^{n}\mathbb{E}[\bar{Y}^2] = n\cdot\mathbb{E}[\bar{Y}^2]$$

($\mathbb{E}[Y_i^2]$: i.d.; $\sum_{i=1}^{n}\mathbb{E}[Y_i\cdot\bar{Y}] = n\bar{Y}$)

$$= \frac{1}{n}\left(n\cdot\mathbb{E}[Y_1^2] - 2\,\mathbb{E}\left[\sum_{i=1}^{n}Y_i\cdot\bar{Y}\right] + n\cdot\mathbb{E}[\bar{Y}^2]\right)$$

$$= \mathbb{E}[Y_1^2] - 2\,\mathbb{E}[\bar{Y}^2] + \mathbb{E}[\bar{Y}^2]$$

$$= \mathbb{E}[Y_1^2] - \mathbb{E}[\bar{Y}^2]$$

$\mathbb{E}[Y_1^2] = \mathrm{Var}[Y_1] + (\mathbb{E}[Y_1])^2 = \sigma^2 + \mu^2$

$\mathbb{E}[\bar{Y}^2] = \mathrm{Var}[\bar{Y}] + (\mathbb{E}[\bar{Y}])^2 = \frac{\sigma^2}{n} + \mu^2$

$$\mathbb{E}[(S')^2] = \sigma^2 + \mu^2 - \left(\frac{\sigma^2}{n} + \mu^2\right) = \left(1 - \frac{1}{n}\right)\sigma^2 = \left(\frac{n-1}{n}\right)\sigma^2$$

$$\Rightarrow \text{bias}\left((S')^2\right) = \mathbb{E}[(S')^2 - \sigma^2] = -\frac{\sigma^2}{n}$$

$$\mathbb{E}\left[(S')^2\cdot\frac{n}{n-1}\right] = \sigma^2$$

$$\mathbb{E}\left[\frac{n}{n-1}\cdot\frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2\right]$$

So, the UNBIASED estimator for σ^2 is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$$

Mean Squared Error.

Def'n. Let $\hat{\theta}$ be an estimator for the parameter θ:

1. the error of $\hat{\theta}$ is $\hat{\theta}-\theta$;
2. the absolute error of $\hat{\theta}$ is $|\hat{\theta}-\theta|$;
3. the squared error of $\hat{\theta}$ is $(\hat{\theta}-\theta)^2$;
4. the mean squared error of $\hat{\theta}$ is

$$\boxed{MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta}-\theta)^2]}$$

Proposition. $\boxed{MSE(\hat{\theta}) = (bias(\hat{\theta}))^2 + Var[\hat{\theta}]}$

$\longrightarrow$:

$$\begin{aligned} MSE(\hat{\theta}) &= \mathbb{E}[(\hat{\theta}-\theta)^2] \\ &= \mathbb{E}[((\hat{\theta}-\mathbb{E}[\hat{\theta}])+(\mathbb{E}[\hat{\theta}]-\theta))^2] \\ &= \mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])^2] + 2\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])\cdot(\mathbb{E}[\hat{\theta}]-\theta)] \\ &\quad + \mathbb{E}[(\mathbb{E}[\hat{\theta}]-\theta)^2] \\ &= Var[\hat{\theta}] + (bias(\hat{\theta}))^2 \end{aligned}$$

($\mathbb{E}[\hat{\theta}]-\theta$ is deterministic)

Focus on $\mathbb{E}[(\hat{\theta}-\mathbb{E}[\hat{\theta}])(\mathbb{E}[\hat{\theta}]-\theta)]$ =

$$= (\mathbb{E}[\hat{\theta}]-\theta)\underbrace{(\mathbb{E}[\hat{\theta}]-\mathbb{E}[\hat{\theta}])}_{=0}$$

□

Def'n. The standard error of $\hat{\theta}$ is

$$\boxed{SD[\hat{\theta}] = \sqrt{Var[\hat{\theta}]}}$$

M378K Introduction to Mathematical Statistics
Problem Set #15
Bias. MSE.

Problem 15.1. Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhall, Scheaffer.
Let $Y_1, , Y_2, Y_3$ be a random sample from $E(\tau)$. Consider the following five estimators of τ:

$$\hat{\theta}_1 = Y_1, \quad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}, \quad \hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \quad \hat{\theta}_4 = 3Y_{(1)}, \quad \hat{\theta}_5 = \bar{Y}.$$

Which ones of these estimators are unbiased? Among the unbiased ones which one has the smallest variance?

→: $\mathbb{E}[\hat{\theta}_1] = \mathbb{E}[Y_1] = \tau$ ✓

$\mathbb{E}[\hat{\theta}_2] = \mathbb{E}\left[\frac{Y_1+Y_2}{2}\right] = \tau$ ✓

$\mathbb{E}[\hat{\theta}_3] = \mathbb{E}\left[\frac{Y_1+2Y_2}{3}\right] = \tau$ ✓?

$\mathbb{E}[\hat{\theta}_4] = \mathbb{E}[3 \cdot Y_{(1)}] = 3\mathbb{E}[Y_{(1)}] = 3 \cdot \frac{\tau}{3} = \tau$ ✓

The cdf $Y_{(1)}$ = ?

$y \in (0,\infty)$: $F_{(1)}(y) = \mathbb{P}[Y_{(1)} \le y] = \mathbb{P}[\min(Y_1, \dots, Y_n) \le y]$

$= 1 - \mathbb{P}[\min(Y_1, \dots, Y_n) > y]$

$= 1 - \mathbb{P}[Y_1 > y, Y_2 > y, \dots, Y_n > y]$

$= 1 - \mathbb{P}[Y_1 > y] \cdot \mathbb{P}[Y_2 > y] \cdots \mathbb{P}[Y_n > y]$

$Y_i \sim E(\tau)$

$= 1 - \left(e^{-\frac{y}{\tau}}\right)^n = 1 - e^{-\frac{ny}{\tau}}$

$= 1 - e^{-\frac{y}{\tau/n}} \Rightarrow Y_{(1)} \sim E(\tau/n)$

$\mathbb{E}[\hat{\theta}_5] = \mathbb{E}[\bar{Y}] = \tau$ ✓