M378K : October 19th, 2026.

Definition 13.1. *Let $Y_1, \dots, Y_n$ be a* **random sample**. *The random sample ordered in an increasing order is called an* order statistic *and denoted by*

$$Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}.$$

Question Write $Y_{(1)}$ as a function of $Y_1, Y_2, \dots, Y_n$.

$Y_{(1)} = \min(Y_1, \dots, Y_n)$

Question Write $Y_{(n)}$ as a function of $Y_1, Y_2, \dots, Y_n$.

$Y_{(n)} = \max(Y_1, \dots, Y_n)$

Problem 13.2. *What is the distribution function of the random variable $Y_{(n)}$?*

→: for $y \in \mathbb{R}$: $F_{(n)}(y) = \mathbb{P}[Y_{(n)} \le y] = \mathbb{P}[\max(Y_1, \dots, Y_n) \le y]$

$= \mathbb{P}[Y_1 \le y, Y_2 \le y, \dots, Y_n \le y]$ (independence)

$= \mathbb{P}[Y_1 \le y] \cdot \mathbb{P}[Y_2 \le y] \cdots \mathbb{P}[Y_n \le y]$ (i.d.)

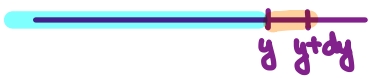
$= (\mathbb{P}[Y_1 \le y])^n = (F_Y(y))^n$

Problem 13.3. *Assume that the random sample comes from a density f_Y. Is the r.v. $Y_{(n)}$ continuous? If so, what is its density $g_{(n)}$?*

→: For all y where F_Y is differentiable:

$$g_{(n)}(y) = \frac{d}{dy} F_{(n)}(y) = \frac{d}{dy}\left((F_Y(y))^n\right) =$$

$$= n \cdot (F_Y(y))^{n-1} \cdot f_Y(y)$$





Problem 13.4. *What is the distribution function of the random variable $Y_{(1)}$?*

→: for $y \in \mathbb{R}$: $F_{(1)}(y) = \mathbb{P}[Y_{(1)} \le y] = \mathbb{P}[\min(Y_1,\dots,Y_n) \le y]$

$= 1 - \mathbb{P}[\min(Y_1,\dots,Y_n) > y] = 1 - \mathbb{P}[Y_1 > y, \dots, Y_n > y]$

(indep) $= 1 - \mathbb{P}[Y_1 > y] \cdot \dots \cdot \mathbb{P}[Y_n > y]$

(i.d.) $= 1 - (\mathbb{P}[Y_1 > y])^n = 1 - (1 - \mathbb{P}[Y_1 \le y])^n$

$= 1 - (1 - F_Y(y))^n$

Problem 13.5. *Assume that the random sample comes from a density f_Y. Is the r.v. $Y_{(1)}$ continuous? If so, what is its density $g_{(1)}$?*

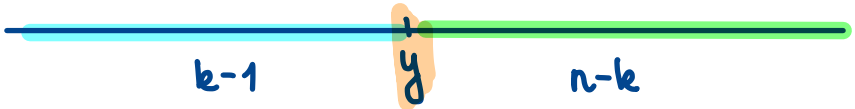
→: for y where F_Y is differentiable:

$g_{(1)}(y) = \frac{d}{dy} F_{(1)}(y) = \frac{d}{dy}\left(1 - (1 - F_Y(y))^n\right)$

$= +n\,(1 - F_Y(y))^{n-1} \cdot (+1) \cdot f_Y(y)$

Theorem 13.2. *Let $Y_1, \dots, Y_n$ be independent, identically distributed random variables with the common cumulative distribution function F_Y and the common probability density function f_Y. Let $Y_{(k)}$ denote the k^{th} order statistic and let $g_{(k)}$ denote its probability density function. Then,*

$$g_{(k)}(y) = \frac{n!}{(k-1)!(n-k)!}(F_Y(y))^{k-1} f_Y(y)(1 - F_Y(y))^{n-k} \quad \text{for all } y \in \mathbb{R}.$$



$n \cdot \binom{n-1}{k-1} \quad (F_Y(y))^{k-1} f_Y(y) \quad (1 - F_Y(y))^{n-k}$

Estimators.

Def'n. The bias of a estimator $\hat{\theta}$ of the parameter θ is defined as

$$\text{bias}(\hat{\theta}) := \mathbb{E}[\hat{\theta} - \theta] = \mathbb{E}\quad - \theta.$$

Notation from books: $\mathbb{E}_\theta(\cdot)$, $\mathbb{E}^\theta(\cdot)$, $\mathbb{E}[\cdots \mid \theta]$

We say that $\hat{\theta}$ is unbiased if

$$\boxed{\text{bias}(\hat{\theta}) = 0 \iff \mathbb{E}[\hat{\theta}] = \theta}$$

for all admissible values of θ

Example. Consider a random sample from $N(\mu, \sigma^2)$ w/ both parameters unknown.

$$\hat{\mu} = \bar{Y} := \frac{1}{n}(Y_1 + \cdots + Y_n)$$

Then, $\mathbb{E}[\hat{\mu}] = \mu$, i.e., $\hat{\mu} = \bar{Y}$ is unbiased for μ.

Example. Let $Y_1, \ldots, Y_n$ be a random sample from $N(\mu_0, \sigma^2)$ w/ μ_0 known and $\sigma > 0$ unknown.

We propose this estimator for the variance σ^2:

$$\boxed{S^2 := \frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_0)^2}$$

Then,

$$\mathbb{E}[S^2] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_0)^2\right]$$

$$= \frac{1}{n}\,\mathbb{E}\left[\sum_{i=1}^{n}(Y_i - \mu_0)^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\underbrace{\mathbb{E}[(Y_i - \mu_0)^2]}_{\sigma^2} = \frac{1}{n}\cdot n \cdot \sigma^2 = \sigma^2$$

$\Rightarrow$ S^2 is unbiased for σ^2

Example. Let $Y_1, \ldots, Y_n$ be a random sample from $N(\mu, \sigma^2)$ w/ both parameters unknown.

Goal: Find a "good" estimator for σ^2!

Propose:

$$(S')^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

Q: Is $(S')^2$ unbiased for σ^2?