**Example 9.14.** *Let the random variable $Y$ have the mgf $m_Y$. Define $X = aY + b$ for some constants $a$ and $b$. Express the mgf $m_X$ of $X$ in terms of $m_Y$, $a$ and $b$.*

→: $m_X(t) = \mathbb{E}[e^{t\cdot X}] = \mathbb{E}[e^{t(aY+b)}] = \mathbb{E}[e^{taY+tb}]$

$= \mathbb{E}[e^{taY} e^{tb}] = e^{tb}\mathbb{E}[e^{taY}] = e^{tb} m_Y(ta)$

**Example 9.15.** *Let $X \sim N(\mu, \sigma^2)$. What is the moment generating function $m_X$ of $X$?*

→: $X = \mu + \sigma \cdot Z$ $\quad Z \sim N(0,1)$

($\mu \leftrightarrow b$, $\sigma \leftrightarrow a$)

$m_X(t) = e^{t\cdot\mu} m_Z(t\cdot\sigma) = e^{t\cdot\mu} \cdot e^{\frac{t^2\sigma^2}{2}} = \boxed{e^{t\mu + \frac{t^2\sigma^2}{2}}}$ □

**Problem 9.2.** *A random variable $Y$ is said to be* lognormal *if there exists a normally distributed random variable $X \sim N(\mu, \sigma^2)$ such that $Y \overset{(d)}{=} e^X$. Express the mean and the variance of the lognormal r.v. $Y$ in terms of the parameters $\mu$ and $\sigma$.*

→: $\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{1\cdot X}] = m_X(1) = e^{\mu + \frac{\sigma^2}{2}}$

$\mathbb{E}[Y^2] = \mathbb{E}[(e^X)^2] = \mathbb{E}[e^{2\cdot X}] = m_X(2) = e^{2\mu + \frac{2^2\sigma^2}{2}} = e^{2(\mu+\sigma^2)}$

$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

**Proposition 9.16.**
1. *If $m_Y$ exists for a certain probability distribution, then it is unique.*
2. *If $m_{Y_1}$ and $m_{Y_2}$ are equal on an interval, then $Y_1 \overset{(d)}{=} Y_2$.*

**Corollary 9.17.** *Let $Y_1$ and $Y_2$ be independent and normally distributed. Define $Y = Y_1 + Y_2$. Then, the distribution of $Y$ is ...*

→: $Y_i \sim N(\mu_i, \sigma_i^2)$, $i = 1,2$

$m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t) = e^{t\mu_1 + \frac{t^2\sigma_1^2}{2}} \cdot e^{t\mu_2 + \frac{t^2\sigma_2^2}{2}}$

$= e^{t(\mu_1+\mu_2) + \frac{t^2(\sigma_1^2+\sigma_2^2)}{2}}$ ($\mu = \mu_1+\mu_2$, $\sigma^2 = \sigma_1^2+\sigma_2^2$)

*Proof.*

$\boxed{Y \sim N(\mu = \mu_1+\mu_2, \sigma^2 = \sigma_1^2+\sigma_2^2)}$ □

□

**Corollary 9.18.** *Let $N_1$ and $N_2$ be independent and Poisson distributed. Define $N = N_1 + N_2$. Then, the distribution of $N$ is …*

$N_i \sim P(\lambda_i)$, $i = 1, 2$.

$\longrightarrow$: $m_N(t) = m_{N_1}(t) \cdot m_{N_2}(t) = e^{\lambda_1 (e^t - 1)} \cdot e^{\lambda_2 (e^t - 1)}$

$= e^{\overbrace{(\lambda_1 + \lambda_2)}^{\lambda} (e^t - 1)} = e^{\lambda (e^t - 1)}$

*Proof.*

$\Rightarrow$ $\boxed{N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)}$ $\square$