The F-Distribution.

Let Y_1 and Y_2 be two **independent** χ^2-distributed r.v.s w/ df = 1.

For both Y_1 and Y_2, the pdf is

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \mathbb{1}_{(0,\infty)}(y)$$

Define $W = \frac{Y_1}{Y_2}$, i.e., $W = g(Y_1, Y_2)$ w/ $g(y_1, y_2) = \frac{y_1}{y_2}$

Goal: Density of W, i.e., f_W!

Start by figuring out the cdf F_W.

- $w \le 0$: $F_W(w) = 0$
- $w > 0$:

$$F_W(w) = \mathbb{P}[W \le w] = \mathbb{P}\left[\frac{Y_1}{Y_2} \le w\right] = \mathbb{P}[Y_1 \le w \cdot Y_2]$$

$$= \int_0^\infty \int_0^{w \cdot y_2} f_{Y_1, Y_2}(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_0^\infty \int_0^{w \cdot y_2} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}} \cdot \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}}\, dy_1\, dy_2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} \underbrace{\int_0^{w \cdot y_2} \frac{1}{\sqrt{2\pi y_1}} e^{-\frac{y_1}{2}}\, dy_1}_{F_{Y_1}(w \cdot y_2)}\, dy_2$$

$$F_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} F_{Y_1}(w \cdot y_2)\, dy_2$$

$$f_W(w) = \frac{d}{dw} \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} F_{Y_1}(w \cdot y_2)\, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} f_{Y_1}(w \cdot y_2) \cdot y_2\, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{\sqrt{2\pi y_2}} e^{-\frac{y_2}{2}} \cdot \frac{1}{\sqrt{2\pi w y_2}} e^{-\frac{w y_2}{2}} \cdot y_2 \, dy_2$$

$$f_W(w) = \int_0^\infty \frac{1}{2\pi} \cdot \frac{1}{\sqrt{w}} \, e^{-\frac{(1+w)}{2} \cdot y_2} \, dy_2$$

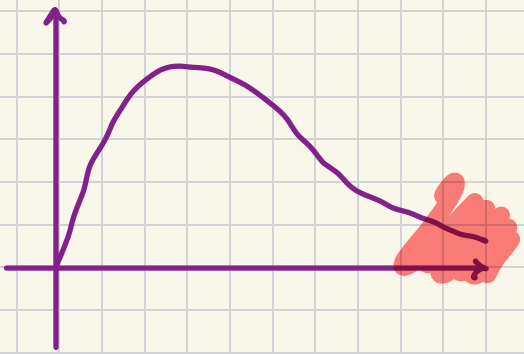
$$f_W(w) = \frac{1}{2\pi\sqrt{w}} \int_0^\infty e^{-\frac{(1+w)}{2} \cdot y_2} \, dy_2 = \frac{2}{1+w}$$

$$f_W(w) = \frac{1}{2\pi\sqrt{w}} \cdot \frac{2}{1+w} = \frac{1}{\pi\sqrt{w}(1+w)} \cdot \mathbb{1}_{(0,\infty)}(w)$$

is the density of $F(1,1)$, i.e., the F-dist'n w/ 1 numerator and 1 denominator df.

In general:

$$F(\nu_1, \nu_2) = \frac{\chi^2(\nu_1)/\nu_1}{\chi^2(\nu_2)/\nu_2}$$



M378K Introduction to Mathematical Statistics
Problem Set #9
Moment generating functions.

Definition 9.1. *The k^{th} moment of a random variable Y* taken about the origin *is defined as $\mathbb{E}[Y^k]$ provided that the expectation exists. We write*

$$\mu_k = \mathbb{E}[Y^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.2. *μ_k is also referred to as the k^{th}* raw moment.

Remark 9.3. *In particular, $\mu_1 = \mu$ happens to be the* **mean** *of the random variable Y.*

Definition 9.4. *The k^{th}* central moment *of a random variable Y is defined as $\mathbb{E}[(Y-\mu)^k]$ provided that the expectation exists. We write*

$$\mu_k^c = \mathbb{E}[(Y-\mu)^k]$$

when there is no ambiguity about the random variable in question.

Remark 9.5. *μ_k is also referred to as the k^{th}* moment of a random variable Y taken about its mean.

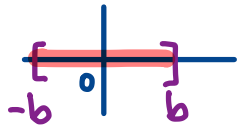
Definition 9.6. *The* moment-generating function (mgf) *m_Y for a random variable Y is defined as*

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

for all t for which the above expectation exists. In fact, we say that the moment-generating function **exists** *if there exists a positive number b such that $m_Y(t)$ is finite for all t such that $|t| \leq b$.*

Problem 9.1. *How much is $m_Y(0)$?*

$m_Y(0) = \mathbb{E}[e^{0\cdot Y}] = 1$



Remark 9.7. *On the choice of terminology ...*

Step 1.

$$\frac{d}{dt} m_Y(t) = ?$$

$\frac{d}{dt} m_Y(t) = \frac{d}{dt}\mathbb{E}[e^{tY}] = \mathbb{E}\left[\frac{d}{dt} e^{tY}\right]$

$= \mathbb{E}[Ye^{tY}]$

Step 2.

$m'_Y(0) =$ ✗ $\mathbb{E}[Ye^{0\cdot Y}] = \mathbb{E}[Y] = \mu$

Step 3.

$\frac{d^2}{dt^2} m_Y(t) = ?$

$\frac{d}{dt}\left(\frac{d}{dt} m_Y(t)\right) = \frac{d}{dt}\left(\mathbb{E}[Ye^{tY}]\right) = \mathbb{E}[Y^2 e^{tY}]$

Step 4.

$m''_Y(0) =$ ✗ $\mathbb{E}[Y^2] = \mu_2$,

i.e., the second moment

Step 5. *What do you suspect the* **generalization** *of the above would be?*

$m^{(k)}(0) = \mathbb{E}[Y^k] = \mu_k$

Theorem 9.8. *If m_Y exists, then for $k \in \mathbb{N}$, we have*

$$m_Y^{(k)}(0) = \mu_k.$$

Example 9.9. *Let $Y \sim b(n = 1, p)$, i.e., let Y model a Bernoulli trial with the probability of success denoted by p. Find m_Y.*

→: $m_Y(t) = \mathbb{E}[e^{tY}] = e^{t\cdot 0}(1-p) + e^{t\cdot 1}\cdot p$

$= (1-p) + pe^t \quad t \in \mathbb{R}$

Proposition 9.10. *Let Y_1 and Y_2 be independent random variables with m.g.f.s denoted by m_{Y_1} and m_{Y_2}. Define $Y = Y_1 + Y_2$. Then, for every t for which both m_{Y_1} and m_{Y_2} are well defined, we have*

$$m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)$$

Proof. By definition:

$$m_Y(t) = \mathbb{E}[e^{tY}]$$

Using $Y = Y_1 + Y_2$, we can subsitute $Y_1 + Y_2$ for Y in the expression above. So,

$$m_Y(t) = \mathbb{E}[e^{t(Y_1+Y_2)}]$$

One of the properties of the exponential function is that $e^{A+B} = e^A \times e^B$. Thus, the above becomes:

$$m_Y(t) = \mathbb{E}[e^{tY_1} \cdot e^{tY_2}]$$

Recall that Y_1 and Y_2 are assumed to be independent random variables. With this in mind, we get:

$$m_Y(t) = \mathbb{E}[e^{tY_1}] \cdot \mathbb{E}[e^{tY_2}]$$

Finally, using the definition of a m.g.f., we have

$$m_Y(t) = m_{Y_1}(t) \cdot m_{Y_2}(t)$$

□

□

Example 9.11. *Let $Y \sim b(n,p)$. What is the moment generating function of Y?*

→: $Y = I_1 + I_2 + \cdots + I_n$ w/ I_j, $j=1..n$, are all $B(p)$ and independent

$m_Y(t) \overset{\text{independence}}{=} m_{I_1}(t) \cdot m_{I_2}(t) \cdots m_{I_n}(t) \overset{\text{identically dist'd}}{=} (m_{I_1}(t))^n = ((1-p)+pe^t)^n$ □

Example 9.12. *Let $N \sim Poisson(\lambda)$. What is the moment generating function m_N of N?*

→: $m_N(t) = \mathbb{E}[e^{t\cdot N}] = \sum_{n=0}^{\infty} e^{t\cdot n} p_N(n) = \sum_{n=0}^{\infty} \left(e^{t\cdot n} e^{-\lambda} \frac{\lambda^n}{n!}\right)$

$= e^{-\lambda} \sum_{n=0}^{\infty} \frac{e^{t\cdot n}\lambda^n}{n!} = e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(e^t\cdot\lambda)^n}{n!} = e^{-\lambda} e^{e^t\cdot\lambda} = e^{\lambda(e^t-1)}$ □

Example 9.13. *Let $Z \sim N(0,1)$. What is the moment generating function m_Z of Z?*

→: $m_Z(t) = \mathbb{E}[e^{t\cdot Z}] = \int_{-\infty}^{\infty} e^{t\cdot z}\cdot\varphi(z)dz = \int_{-\infty}^{\infty} e^{t\cdot z}\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}+t\cdot z-\frac{t^2}{2}} \cdot e^{\frac{t^2}{2}} dz = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz$

$= 1$ (density of $N(t,\sigma^2=1)$)

$m_Z(t) = e^{t^2/2}$