The University of Texas at Austin
Homework assignment 7

Introduction to Mathematical Statistics

March 07, 2026

Instructions: Provide your complete solution to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

Moment generating functions

Problem 7.1. ($3 \times 7 = 21$ points) Identify the distributions with the following mgfs. If the distribution has a name, give the name and the parameters. If it does not, give the pdf or the pmf (table).

- $\frac{2}{2-t}$

- e^{2e^t-2}

- $e^{t(t-2)}$

- $\left(3-2e^t\right)^{-1}$

- $\frac{1}{9}+\frac{4}{9}e^t+\frac{4}{9}e^{2t}$

- $\frac{1}{t(1-e^{-t})}$

- $\frac{1}{4}\left(e^{4t}+3e^{-t}\right)$

Solution. All but the last one are named distribution and can be found in the table/notes:

1. $E(1/2)$.
2. $P(2)$,
3. Since $t(t-2)=(-2)t+\frac{1}{2}\left(\sqrt{2}\right)^2t^2$, this is $N\left(-2,\sqrt{2}\right)$.
4. Since $\frac{1}{3-2e^t}=\frac{\frac{1}{3}}{1-\frac{2}{3}e^t}$, this is $g\left(\frac{1}{3}\right)$.
5. $B\left(2,\frac{2}{3}\right)$.
6. $U(-1,0)$
7. The last one corresponds to the discrete distribution with support $\{-1,4\}$ with the pmf $p(-1)=\frac{3}{4},p(4)=\frac{1}{4}$.

Problem 7.2. (30 points) Solve **Problem 7.6.11** from the *Lecture notes*.

Solution.

1. $\Gamma(1) = \int_0^\infty e^{-y}\,\mathrm{d}y = 1$. Using the fact that $\lim_{y\to 0} ye^{-y} = 0$ and $\lim_{y\to\infty} ye^{-y} = 0$, for $n = 2$, we have

$$\Gamma(2) = \int_0^\infty ye^{-y}\,\mathrm{d}y = (-ye^{-y})\,|_0^\infty + \int_0^\infty e^{-y}\,dy = (0-0) + \int_0^\infty e^{-y}\,\mathrm{d}y = \Gamma(1) = 1.$$

   Similarly,

$$\Gamma(3) = \int_0^\infty y^2e^{-y}\,\mathrm{d}y = (-y^2e^{-y})\,|_0^\infty + \int_0^\infty 2ye^{-y}\,dy = (0-0) + 2\int_0^\infty ye^{-y}\,\mathrm{d}y = 2\Gamma(2) = 2,$$
$$\Gamma(4) = \int_0^\infty y^3e^{-y}\,\mathrm{d}y = (-y^3e^{-y})\,|_0^\infty + \int_0^\infty 3y^2e^{-y}\,dy = (0-0) + 3\int_0^\infty y^2e^{-y}\,\mathrm{d}y = 3\Gamma(3) = 6.$$

   In general,

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty y^ne^{-y}\,\mathrm{d}y = (-y^ne^{-y})\,|_0^\infty + \int_0^\infty ny^{n-1}e^{-y}\,dy = (0-0) + n\int_0^\infty y^{n-1}e^{-y}\,\mathrm{d}y \\ &= n\Gamma(n).\end{aligned}$$

   From here it follows immediately that $\Gamma(n) = (n-1)!$.

2. $c = 1/d$, where $d = \int_0^\infty y^{k-1}e^{-y/\tau}\,\mathrm{d}y$. We change the variables in the integral by setting $z = y/\tau$, so that $dy = \tau\,\mathrm{d}z$ and

$$d = \int_0^\infty \tau^{k-1}z^{k-1}e^{-z}\tau\,\mathrm{d}z = \tau^k\int_0^\infty z^{k-1}e^{-z}\,\mathrm{d}z = \tau^k\Gamma(k).$$

3. For $t < 1/\tau$, we have

$$\begin{aligned}m(t) &= \int_0^\infty e^{ty}\frac{1}{\tau^k\Gamma(k)}y^{k-1}e^{-y/\tau}\,\mathrm{d}y = [z = (1/\tau - t)y] \\ &= \frac{1}{\tau^k\Gamma(k)}\int_0^\infty e^{-z}z^{k-1}(1/\tau - t)^{-k}\,\mathrm{d}z \\ &= (1-\tau t)^{-k}\frac{1}{\Gamma(k)}\int_0^\infty e^{-z}z^{k-1}\,\mathrm{d}z = (1-\tau t)^{-k},\end{aligned}$$

   so, yes, our guess was correct:

$$f_Y(y) = \frac{1}{\tau^k\Gamma(k)}y^{k-1}e^{-y/\tau}1_{\{y>0\}}$$

   is indeed the pdf of the gamma distribution with parameters k and τ.

Problem 7.3. (18 points) *Source: "Mathematical Statistics with Applications" by Wackerly, Mendenhal, Scheaffer.*
Suppose that a random variable Y has a probability density function given by

$$f_Y(y) = \kappa y^3 e^{-y/2} 1_{(0,\infty)}(y)$$

- (5 points) Find the value of κ that makes $f_Y(y)$ a density function.
- (3 points) Does Y have a χ^2 −distribution? If so, how many degrees of freedom?
- (5 points) What are the mean and standard deviation of Y?
- (5 points) (Extra credit) Using **R**, find the probability that Y lies within 2 standard deviations of its mean?

Solution.

- Using the findings of the previous problem, we first realise that Y is Gamma distributed with $k = 4$ and $\tau = 2$ in our usual parameterization. Then, we conclude that

  $$\kappa^{-1} = \tau^k \Gamma(k) = 2^4 \Gamma(4) = 2^4 \cdot 3! = 96.$$

  So, $\kappa = 1/96$.
- Since $\tau = 2$, this is a χ^2 −distribution with $2\kappa = 8$ degrees of freedom.
- The mean is $\mathbb{E}[Y] = k\tau = 8$ and the standard deviation is $SD[Y] = \tau\sqrt{k} = 4$.
- We are looking for the probability

  $$\mathbb{P}[0 \leq Y \leq 16].$$

  In f R, we can use ‘pchisq(16, df=8)‘ to get 0.9576199.