M362K: March 25th, 2024.

Section 3.2.

Expectation.

→ expected value, mean (very rarely: average)

Inspiration: N tickets

N_1 w/ payoff x_1

N_2 w/ payoff x_2

$\vdots$

N_m w/ payoff x_m

Money in $\longrightarrow$ $\boxed{\text{Lottery}}$ $\longrightarrow$ Money out

$$N \cdot P \;=\; N_1 \cdot x_1 + N_2 x_2 + \cdots + N_m \cdot x_m$$

price of one ticket

if all is "fair" ("break-even price")

$$P = \frac{N_1}{N} x_1 + \frac{N_2}{N} x_2 + \cdots + \frac{N_m}{N} \cdot x_m$$

for every $i = 1..m$, p_i ... the probability of getting a ticket w/ payoff x_i (if all tickets are equally likely)

$$\boxed{P = \sum_{i=1}^{m} p_i \cdot x_i}$$

... a weighted average of payoffs

Note: We already talked about the "mean" of the binomial distribution Binomial(n, p).
We had

$$\boxed{\mu = np}$$

Def'n. The mean (expectation, expected value) of a probability distribution $P(x)$ over a finite set Ω of values x is

$$\boxed{\mu = \sum_{\text{all } x} P(x) \cdot x}$$

Def'n. Let X be a random variable on a finite Ω. The expectation (expected value, mean) of X is denoted by $\mathbb{E}[X]$ and it is defined as

$$\boxed{\mathbb{E}[X] = \sum_{i=1}^{m} x_i \, p_X(x_i)}$$

where $\{x_1, \dots, x_m\}$ is the support of X

and p_X is the pmf of X.

Addition Rules for Expectation.

Let X and Y be r.v.s on the same Ω, and let $\mathcal{K}$ be any constant (a real number).

Then,

- $\mathbb{E}[\mathcal{K} \cdot X] = \mathcal{K} \cdot \mathbb{E}[X]$ homogeneity
- $\mathbb{E}[X + \mathcal{K}] = \mathbb{E}[X] + \mathcal{K}$ translativity
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ additivity

$$\boxed{\mathbb{E}[\mathcal{K} \cdot X + Y] = \mathcal{K} \cdot \mathbb{E}[X] + \mathbb{E}[Y] \qquad \text{linearity}}$$

More generally, for random variables $X_1, X_2, \dots, X_n$ on the same Ω and constants $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n$, we have

$$\boxed{\mathbb{E}[\mathcal{K}_1 X_1 + \mathcal{K}_2 X_2 + \dots + \mathcal{K}_n X_n] = \mathcal{K}_1 \mathbb{E}[X_1] + \mathcal{K}_2 \mathbb{E}[X_2] + \dots + \mathcal{K}_n \mathbb{E}[X_n]}.$$

Example. Indicator Random Variables.

A ... an event in Ω $(A \subseteq \Omega)$

Define:
$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

$\mathbb{E}[\mathbb{I}_A] = ?$

1° $\text{Support}(\mathbb{I}_A) = \{0, 1\}$

2° pmf = ? $\quad p_{\mathbb{I}_A}(0) = \mathbb{P}[A^c]$, $p_{\mathbb{I}_A}(1) = \mathbb{P}[A]$

3° Use the def'n of $\mathbb{E}$:

$$\mathbb{E}[\mathbb{I}_A] = \cancel{0 \cdot \mathbb{P}[A^c]} + 1 \cdot \mathbb{P}[A]$$

$$\boxed{\mathbb{E}[\mathbb{I}_A] = \mathbb{P}[A]}$$

Example. $\boxed{X \sim \text{Bernoulli}(p)}$

$$\mathbb{E}[X] = 0 \cdot (\underbrace{1-p}_{q}) + 1 \cdot p = p$$

$$\boxed{\mathbb{E}[X] = p}$$

Example. $X \sim \text{Binomial}(n, p)$

$\mathbb{E}[X] = ?$

Method I. By def'n

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot p_X(k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = \dots$$

Method II. X ... # of successes in i.i.d. Bernoulli trials

Introduce n indicator r.v.s (one for each trial)

$$j = 1 \dots n: \quad I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial success} \\ 0 & \text{if } j^{th} \text{ trial is failure} \end{cases}$$

$$\boxed{X = I_1 + I_2 + \dots + I_n}$$

$$\underline{\mathbb{E}[X]} = \mathbb{E}[I_1 + I_2 + \cdots + I_n]$$ linearity of $\mathbb{E}$

$$= \underbrace{\mathbb{E}[I_1]}_{p} + \underbrace{\mathbb{E}[I_2]}_{p} + \cdots + \underbrace{\mathbb{E}[I_n]}_{p}$$

$$= \underline{n \cdot p}$$

□

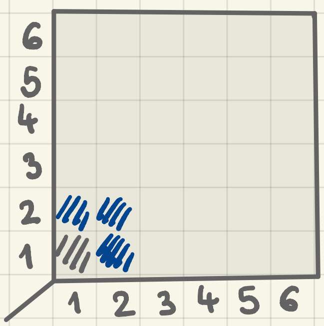
> Whenever two r.v.s are identically dist'n, they have the same expectation.

Example. We toss two dice independently. Let X denote the maximum outcome on the two dice. What is the expectation of X?

→: $\text{Support}(X) = \{1, 2, 3, 4, 5, 6\}$

Find the pmf $p_X(i)$ for $i = 1, 2, 3, 4, 5, 6$

fix i: $\boxed{p_X(i) = \mathbb{P}[X = i]}$



$$p_X(1) = \frac{1}{36}$$

$$p_X(2) = \frac{3}{36}$$

$\vdots$

		# of element. outcomes
max equal to i	both dice are i	1
		$+$
	one is i, the other one is smaller	$(i-1) + (i-1)$
		$2i-1$

$$\mathbb{E}[X] = \sum_{i=1}^{6} i \cdot p_X(i) = \sum_{i=1}^{6} i \cdot \frac{2i-1}{36} = \cdots = \frac{161}{36}$$

□