M362K: March 22nd, 2024.

Independent Pairs.

Def'n. Random variables X and Y are independent if

$$\boxed{\begin{gathered}\mathbb{P}[X\le a, Y\le b] = \mathbb{P}[X\le a]\cdot\mathbb{P}[Y\le b] = F_X(a)\cdot F_Y(b)\\ \text{for all } a,b\in\mathbb{R}\end{gathered}}$$

Proposition. For X and Y independent,

$$\boxed{\begin{gathered}\mathbb{P}[X\in A, Y\in B] = \mathbb{P}[X\in A]\cdot\mathbb{P}[Y\in B]\\ \text{for all "nice" } A,B\subseteq\mathbb{R}\end{gathered}}$$

Corollary. For X and Y on a finite Ω w/ the joint pmf $p_{X,Y}(x,y)$ and the marginal pmf $p_X(x)$ and $p_Y(y)$, we have that X and Y are independent

iff

$$\boxed{\begin{gathered}p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)\\ \text{for all } (x,y)\end{gathered}} \quad (*)$$

Q: What if we need to say if X and Y are independent?

$\longrightarrow$: Yes. Verify that $(*)$ is true for all pairs (x,j)

No. Find one single pair (x,y) such that $(*)$ is not true.

Example [cont'd]. $\Omega = \{HHH, HTT, \ldots\ldots, TTT\}$

X... # of Hs in the first two tosses

Y... # of Ts in the last two tosses

Q: Are X and Y independent?

$\longrightarrow$: $p_{X,Y}(0,0) = 0$ $\neq$ $p_X(0)\cdot p_Y(0) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$ $\boxed{\text{Not independent.}}$

Example. Sum of independent random variables.

Assume $\text{Support}(X) = \{0, 1, \dots, n\}$

$\text{Support}(Y) = \{0, 1, \dots, m\}$

Let their pmfs be: $p_X(0), p_X(1), \dots, p_X(n)$

$p_Y(0), p_Y(1), \dots, p_Y(m)$

Define: $\boxed{S = X+Y}$

$\text{Support}(S) = \{0, 1, \dots, n+m\}$

for $k = 0, 1, \dots, n+m$:

$$p_S(k) = \mathbb{P}[S=k] = \mathbb{P}[X+Y=k]$$

$$= \sum_{j=0}^{k} \mathbb{P}[X+Y=k, X=j]$$

$\boxed{\text{Law of Total Probability}}$

$$= \sum_{j=0}^{k} \mathbb{P}[j+Y=k, X=j]$$

$$= \sum_{j=0}^{k} \mathbb{P}[Y=k-j, X=j]$$

Independence of X and Y

$$= \sum_{j=0}^{k} \mathbb{P}[X=j]\cdot\mathbb{P}[Y=k-j] = \sum_{j=0}^{k} p_X(j)\cdot p_Y(k-j)$$

Compare to: • Convolution (Power Series)

Example. $X \sim \text{Binomial}(n, p)$, $Y \sim \text{Binomial}(m, p)$ } independent

the same success probability

Define $\boxed{S = X + Y}$

$\text{Support}(S) = \{0, 1, \dots, n+m\}$

Probabilistic Approach: $S \sim \text{Binomial}(n+m, p)$

for $k = 0, 1, \dots, n+m$

$$p_S(k) = \boxed{\binom{n+m}{k}} p^k (1-p)^{n+m-k}$$

Brute Force: From the previous example:

$$p_S(k) = \sum_{j=0}^{k} p_X(j) \cdot p_Y(k-j) = \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j} \binom{m}{k-j} p^{k-j} (1-p)^{m-k+j}$$

$$= \boxed{\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}} p^k (1-p)^{m+n-k}$$

$$\boxed{\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}}$$

Vandermonde Convolution.