M362K: March 18^{th}, 2024.

Section 3.1 (cont'd).

Random Pairs.

Still on a finite Ω (but not for long).

Def'n. A random pair (X,Y) is a function from Ω to $\mathbb{R}^2$, i.e.,

$$\boxed{(X(\omega),Y(\omega)) = (X,Y)(\omega)} \quad \text{for every } \omega\in\Omega$$

Example. Ω ... your probability class

ω ... the students in the class

Let X is the students' height.
Let Y is the students' weight.

(X,Y) ... a pair of the students' physical characteristics

We consider the joint outcome of (X,Y) and their joint distribution.

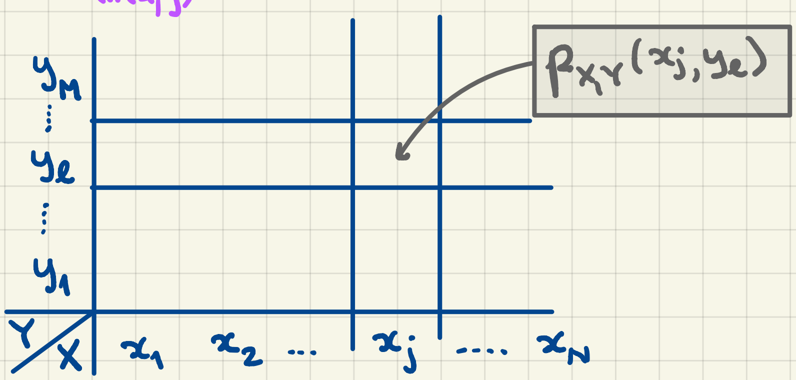
More precisely, we are interested in the probabilities of events such as

$$\{X=x\} \text{ and } \{Y=y\}, \text{ i.e.,}$$
$$\{X=x, Y=y\}$$

The joint pmf is $\boxed{p_{X,Y}(x,y) = \mathbb{P}[X=x, Y=y]}$ for all (x,y) in the support of (X,Y)

Note:

- $0 \leq p_{X,Y}(x,y) \leq 1$
- $\sum_{\text{all}(x,y)} p_{X,Y}(x,y) = 1$ (Law of Total 1).



Example.

Let X and Y be the first and second draws made @ random from a box containing three tickets w/ numbers 1, 2, and 3 on them.

(i) with replacement

p_X ⟵ marginal probabilities

Y \ X	1	2	3	p_Y
3	1/9	1/9	1/9	1/3
2	1/9	1/9	1/9	1/3
1	1/9	1/9	1/9	1/3
p_X	1/3	1/3	1/3	

(ii) without replacement

Y \ X	1	2	3	p_Y
3	1/6	1/6	0	1/3
2	1/6	0	1/6	1/3
1	0	1/6	1/6	1/3
p_X	1/3	1/3	1/3	

Def'n. Let $p_{X,Y}$ be the joint pmf of (X,Y).

The marginal pmf of X is given by:

$$\boxed{p_X(x) = \sum_{\substack{\text{all } y \\ \text{in the} \\ \text{Support}(Y)}} p_{X,Y}(x,y)} \quad \text{for } x \in \text{Support}(X)$$

The marginal pmf of Y is

$$\boxed{p_Y(y) = \sum_{\text{all } x} p_{X,Y}(x,y)} \quad \text{for } y \in \text{Support}(Y)$$

Def'n. (Naive)

We say that r.v. X and Y are **identically distributed** (or have the same distribution) if:

(i) they have the same support

and

(ii) for every value v in their support,

$$\boxed{\mathbb{P}[X=v]=\mathbb{P}[Y=v]}$$

More generally, for any $a \leq b$, we have

$$\boxed{\mathbb{P}[a<X\leq b]=\mathbb{P}[a<Y\leq b]}$$

Equivalently, $\boxed{F_X(v)=F_Y(v) \text{ for all } v\in\mathbb{R}}$

Def'n. Random variables X and Y are said to be **equal** if

$$\boxed{\mathbb{P}[X=Y]=1}$$

Implicitly, this means that X and Y are on the same Ω.

We write: $\boxed{X=Y}$

Example. $\Omega=\{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

# of Hs in first two tosses	X	2	1	1	2	1	1	0	0
# of Ts in last two tosses	Y	0	0	1	1	2	1	1	2

Q: What's the joint pmf?

$Y \backslash X$	0	1	2	
2	1/8	1/8	0	
1	1/8	1/4	1/8	1/2
0	0	1/8	1/8	

Given that $Y=0$, what is the distribution of X?

The possible values of X are 1 and 2.

$$\mathbb{P}[X=1 \mid Y=0] = \frac{1}{2}$$

$$\mathbb{P}[X=2 \mid Y=0] = \frac{1}{2}$$

In general, (X,Y) are a random pair

$p_{X,Y}$ is their joint pmf

p_X, p_Y are marginal pmfs for X and Y, resp.

Given that $Y=y$, what is the probability that $X=x$?

$$p_{X|Y}(x|y) := \mathbb{P}[X=x \mid Y=y] = \frac{\mathbb{P}[X=x, Y=y]}{\mathbb{P}[Y=y]} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

the conditional pmf of X given $Y=y$

In our example:

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{1/8}{1/2} = 1/4 = p_{X|Y}(2|1)$$

$$p_{X|Y}(1|1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{1/4}{1/2} = 1/2$$