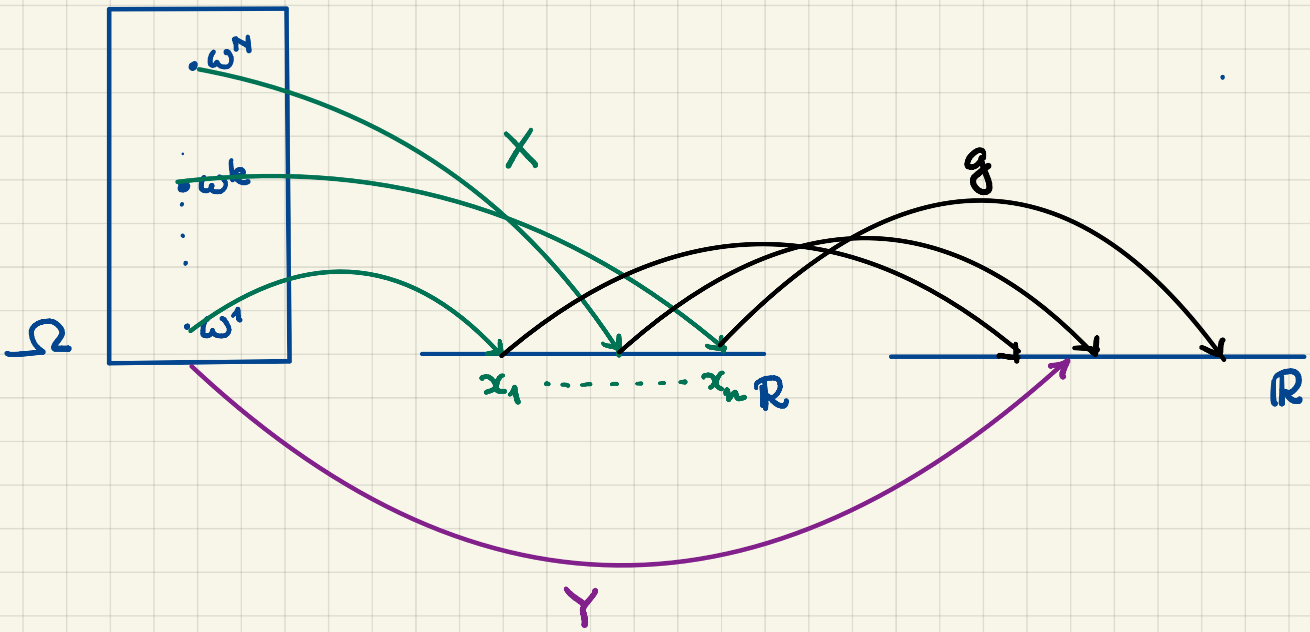
M362K: March 8th, 2024.

# More on Functions of Random Variables.




$\boxed{Y = g(X)}$ is also a random variable

e.g., $g(x) = x^2 \Rightarrow$ the new r.v. is $g(X) = X^2$
$g(x) = \exp(x) \Rightarrow$ the new r.v. is $g(X) = \exp(X)$

We can get the dist'n of $Y := g(X)$ from the dist'n of $X$:

$$\mathbb{P}[Y=y] = \mathbb{P}[g(X)=y] = \sum_{g(x)=y} \mathbb{P}[X=x]$$

Example. A roll of an ordinary fair die.

$X$... the result of the roll

$$Y = \begin{cases} 1 & \text{if the result is even} \\ 0 & \text{if the result is odd} \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & \text{if } X(\omega) \text{ is even} \\ 0 & \text{if } X(\omega) \text{ is odd} \end{cases},$$

i.e., $Y = g(X)$

$$\text{w/ } g(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$$

$\text{Support}(Y) = \{0,1\}$

the pmf of $Y$: $p_Y(0) = \mathbb{P}[Y=0] = \mathbb{P}[X \text{ is } 1 \text{ or } 3 \text{ or } 5]$

$= \mathbb{P}[X=1] + \mathbb{P}[X=3] + \mathbb{P}[X=5] = \frac{1}{2}$

$p_Y(1) = \mathbb{P}[Y=1] = \mathbb{P}[X \text{ is } 2 \text{ or } 4 \text{ or } 6]$

$= \mathbb{P}[X=2] + \mathbb{P}[X=4] + \mathbb{P}[X=6] = \frac{1}{2}$

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## Section 4.5. (Part I).

### Cumulative Distribution Function (cdf).

Consider all $x \in \mathbb{R}$ and look at

$$\boxed{F_X(x) := \mathbb{P}[X \leq x]} \quad \text{for all } x \in \mathbb{R}$$

We obtained a function $\boxed{F_X : \mathbb{R} \longrightarrow [0,1]}$

This is called the cumulative distribution function (cdf) of the random variable $X$.

**Caveat:** "DISTRIBUTION of $X$" refers to the assignment of probabilities of the type $\boxed{\mathbb{P}[X \in A] \text{ for } A \subseteq \mathbb{R}}$

BUT "the CUMULATIVE DISTRIBUTION FTION of $X$" is exactly the function $F_X$

**Usage:** Knowing $F_X$, we can find the probabilities that $X$ falls within any interval $(a,b]$.

i.e., $\mathbb{P}[X \in (a,b]] = \mathbb{P}[a < X \leq b] =$

$$= \mathbb{P}[X \le b] - \mathbb{P}[X \le a]$$

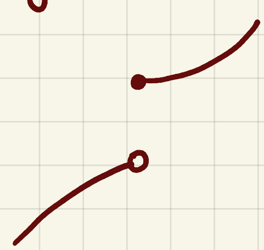
$$\overset{\text{by def'n of } F_X}{=} F_X(b) - F_X(a)$$

$\Rightarrow$ for $a \le b$, $F_X(b) = F_X(a) + \underbrace{\mathbb{P}[X \in (a,b]]}_{\ge 0} \ge F_X(a)$

$\Rightarrow$ $F_X$ is nondecreasing

Other properties:

- $F_X(-\infty) = 0$
- $F_X(+\infty) = 1$
- $F_X$ is right continuous w/ left limits everywhere

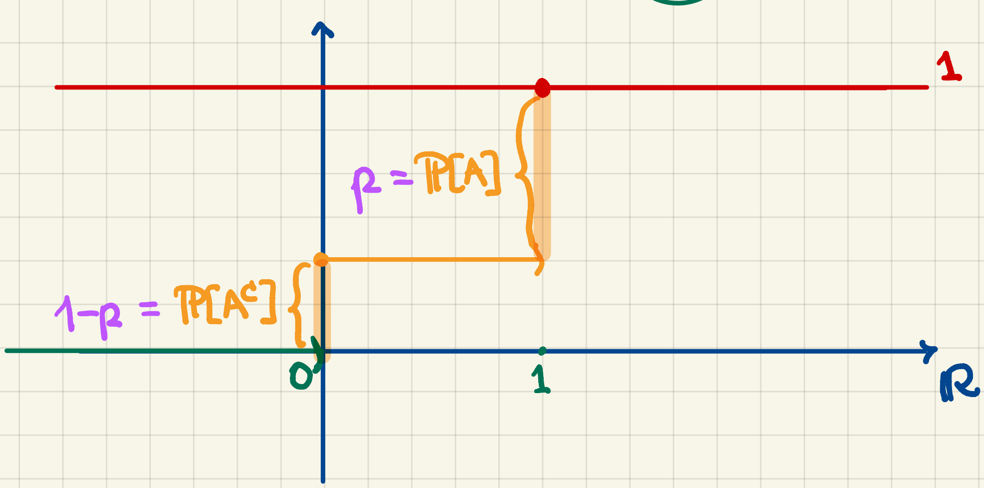


Example. Indicator Random Variable.

For an event $A \subseteq \Omega$, we define

$$\mathbb{I}_A = \begin{cases} 1 & \text{if } A \text{ happened} \\ 0 & \text{if } A \text{ did not happen} \end{cases}$$

Q: What is the form of $F_{\mathbb{I}_A}$?




This is an example of a STEP FUNCTION.

For $x \in (-\infty, 0)$ (i.e., $x<0$)

$$F_{\mathbb{I}_A}(x) = \mathbb{P}[\mathbb{I}_A \leq x] \leq \mathbb{P}[\mathbb{I}_A < 0] = 0$$

For $x=0$: $F_{\mathbb{I}_A}(x) = \mathbb{P}[\mathbb{I}_A \leq 0] = \mathbb{P}[\mathbb{I}_A = 0] = \mathbb{P}[A^c]$

For $x \in (0,1)$: $F_{\mathbb{I}_A}(x) = \mathbb{P}[\mathbb{I}_A \leq x] = \mathbb{P}[\mathbb{I}_A = 0] = \mathbb{P}[A^c]$

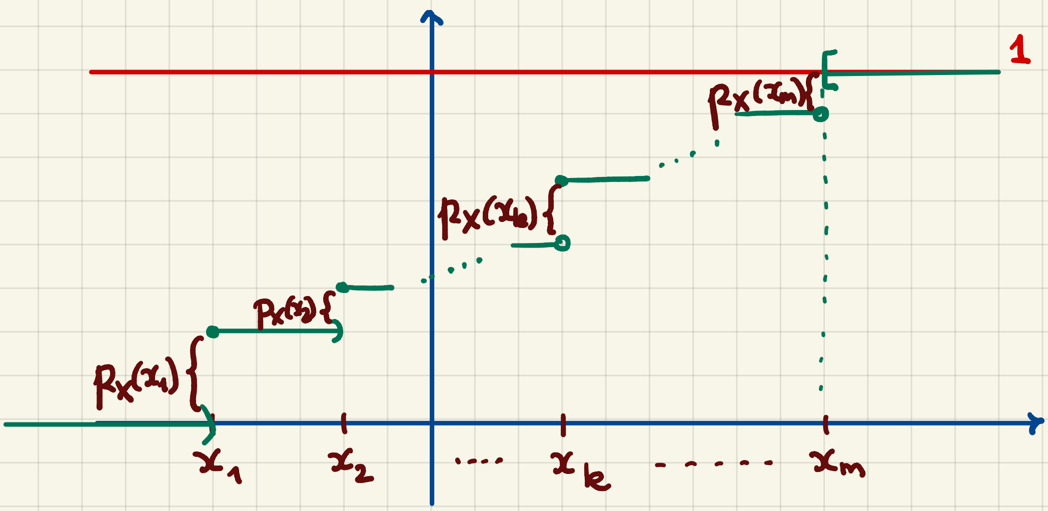
For $x=1$: $F_{\mathbb{I}_A}(x) = \mathbb{P}[\mathbb{I}_A \leq 1] = \mathbb{P}[\mathbb{I}_A = 0] + \mathbb{P}[\mathbb{I}_A = 1] = 1$

For Bernoulli($p$), it's analogous to $\mathbb{P}[A]=p$

In general, let $X$ be a random variable w/

$$\text{Support}(X) = \{\underline{x_1, x_2, \dots, x_m}\}$$

Assume : $x_1 < x_2 < \cdots < x_m$




<u>sizes of jumps</u>
$$\begin{cases} p_X(x_1) = \mathbb{P}[X=x_1] \\ p_X(x_2) = \mathbb{P}[X=x_2] \\ \vdots \\ p_X(x_k) = \mathbb{P}[X=x_k] \\ \vdots \\ p_X(x_m) = \mathbb{P}[X=x_m] \end{cases}$$

For all $y \in \mathbb{R}$: $\mathbb{P}[X=y] = F_X(y) - F_X(y-)$

left limit of $F_X$ @ the point $y$

$$\boxed{F_X(x) = \sum_{y \leq x} p_X(y)}$$