M362K: March 4th, 2024.

Section 3.1.

Random Variables

... on a *finite* Ω (for now).

Example. Consider an experiment consisting of 3 independent tosses of a fair coin.
What is the outcome space Ω corresponding to this experiment?

$$\Omega = \{HHH, THH, HTH, HHT, TTH, THT, HTT, TTT\}$$

Q: How do we model the number of Hs in the first two cointosses?

$$\begin{array}{ccccccccc} \Omega = \{ & HHH, & THH, & HTH, & HHT, & TTH, & THT, & HTT, & TTT\} \\ X & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \end{array}$$

$\leftarrow$ Now we have a function from Ω into $\mathbb{R}$.

$\boxed{\begin{array}{l}\textbf{Def'n.} \text{ Let } \Omega \text{ be a finite outcome space.} \\ \text{Any function } X: \Omega \to \mathbb{R} \text{ is called a RANDOM VARIABLE.}\end{array}}$

Note: By convention, random variables are usually denoted by the capital letters of the latin alphabet.

$$\begin{array}{c} \Omega = \{\omega^1, \omega^2, \dots, \omega^N\} \\ X \downarrow \quad \downarrow \quad \searrow \swarrow \end{array}$$

$\text{Support}(X) = \{x_1, x_2, \dots, x_m\} \subseteq \mathbb{R}$ is every possible value X can take

Def'n. The probability (mass) function (pmf) of X is defined as follows:

$$\boxed{p_X(k) := \mathbb{P}[X = x_k] \quad \text{for all } k = 1, \dots, m}$$

Example [cont'd].

Support(X) = $\{0, 1, 2\}$

It's usually convenient to place the **pmf** in a table:

k	0	1	2
$p_X(k)$	$1/4$	$1/2$	$1/4$

In general:

$$\boxed{\sum_{k=1}^{m} p_X(x_k) = 1}$$

Example. Bernoulli distribution w/ success probability p.

$$\Omega = \{0, 1\}$$
$$X \downarrow \quad \downarrow$$
$$\text{Support}(X) = \{0, 1\}$$

The **pmf** of X:

$$p_X(0) = \mathbb{P}[X = 0] = 1 - p$$
$$p_X(1) = \mathbb{P}[X = 1] = p$$

We write: $X \sim \text{Bernoulli}(p)$

OR

$X \sim b(p)$

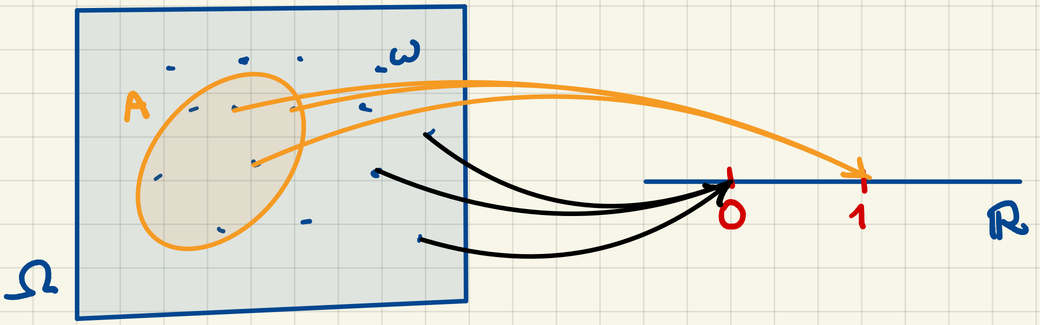
We say: X has the Bernoulli dist'n w/ parameter p

OR

X is Bernoulli dist'd w/ parameter p.

Example.

Consider an event A on an outcome space Ω.



We define an indicator random variable of the event A.

$$\omega \in \Omega: \quad \mathbb{I}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$\text{Support}(\mathbb{I}_A) = \{0, 1\}$

The pmf of $\mathbb{I}_A$:

k	0	1
$p_{\mathbb{I}_A}(k)$	$1-\mathbb{P}[A]$	$\mathbb{P}[A]$

In other words: $\boxed{\mathbb{I}_A \sim \text{Bernoulli}(p = \mathbb{P}[A])}$

Example. Binomial w/ n trials and success probability p.

It's sufficient: $\Omega = \{0, 1, \dots, n-1, n\}$

$$\downarrow X$$

$\text{Support}(X) = \{0, 1, \dots, n-1, n\}$

Its pmf is:

$$\boxed{\text{for all } k = 0, \dots n: \quad p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

We write: $X \sim \text{Binomial}(n, p)$ or $X \sim b(n, p)$

We say: X is binomial w/ parameters n and p.

Example. Consider rolling a fair ordinary die.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Let the random variable X be the value on the upturned face.

$X(1) = 1$
$X(2) = 2$
$X(3) = 3$
$X(4) = 4$
$X(5) = 5$
$X(6) = 6$

$$\boxed{p_X(k) = \frac{1}{6} \quad k = 1, \dots, 6}$$

Q: What's the probability that X is even?

$\longrightarrow$: $\mathbb{P}[X \text{ is even}] = \mathbb{P}[X \in \{2,4,6\}] = \frac{1}{2}$

Let Y be the random variable defined as:

$$Y = \begin{cases} 1 & \text{if the result is prime} \\ 0 & \text{otherwise} \end{cases}$$

$Y(1) = 0$
$Y(2) = 1$
$Y(3) = 1$
$Y(4) = 0$
$Y(5) = 1$
$Y(6) = 0$

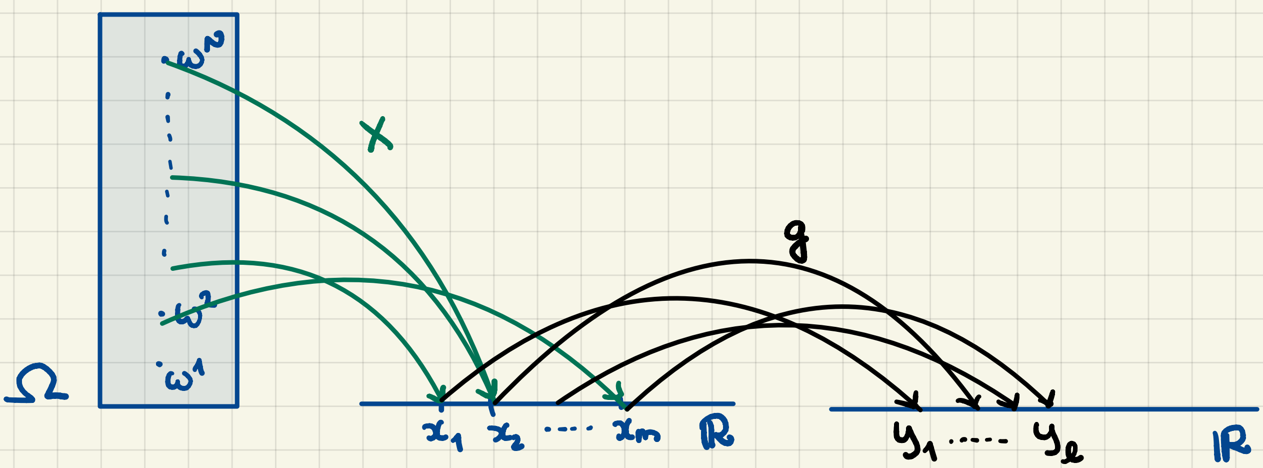
$\text{Support}(Y) = \text{Range}(Y) = \{0,1\}$

$p_Y(0) = \frac{1}{2}$

$p_Y(1) = \frac{1}{2}$

$\square$

Functions of random variables.



$$\boxed{g(X(\omega)) = g(X)(\omega)}$$

A function of a random variable is (again) a random variable.

We assign it a new "Label", e.g., $\boxed{Y = g(X)}$

We can look @ its distribution ☺