M362K: March 1st, 2024.

Section 2.4.

Poisson Approximation

$\boxed{n \text{ large}}$

the successes are RARE EVENTS $p \ll 1$

mean : "$n \cdot p = \mu$"

standard deviation: $\sigma = \sqrt{n \cdot p \underbrace{(1-p)}_{\approx 1}} \approx \sqrt{n \cdot p} = \sqrt{\mu}$

$\boxed{\text{normal approximation BAD}}$

alternative: Poisson approximation

$\boxed{n \to \infty \quad \text{and} \quad p = \frac{\mu}{n} \longrightarrow 0}$

$$\boxed{\mathbb{P}[k \text{ successes}] \approx e^{-\mu} \cdot \frac{\mu^k}{k!} \quad \text{w/ } \mu = np \quad \text{for all } k = 0, 1, 2, \ldots}$$

Properties of the Poisson Dist'n.

$\boxed{p_k := e^{-\mu} \cdot \frac{\mu^k}{k!}}$ for all $k = 0, 1, 2, \ldots$

Clearly: $\boxed{p_k > 0}$ for all k

$$\sum_{k=0}^{+\infty} p_k \overset{\cancel{?}}{=} 1$$

$$\sum_{k=0}^{+\infty} e^{-\mu} \frac{\mu^k}{k!} \overset{\cancel{?}}{=} 1$$

$$e^{-\mu} \boxed{\sum_{k=0}^{+\infty} \frac{\mu^k}{k!}} \overset{\cancel{?}}{=} 1$$

Taylor Approximation for e^{μ}

This is our second distribution on a countable outcome space.

$$\Omega = \{0, 1, 2, \dots\} = \mathbb{N}_0$$

w/ $\boxed{p_k = e^{-\mu} \cdot \frac{\mu^k}{k!}}$ for $k \in \Omega$

Problem. A 13-sided die is thrown 169 times (each of the numbers 1, 2, ..., 13 is equally likely on each throw).
Every time we get 13, we get to pick a card from an ordinary deck. If the picked card in an ace, we get a Pokémon plushie.
What is the probability that we get @ least one plushie?

$\longrightarrow$: probability of success: $p = \underbrace{\frac{1}{13}}_{\text{die}} \cdot \underbrace{\frac{1}{13}}_{\substack{\text{4 aces} \\ \text{among} \\ \text{42 cards}}} = \frac{1}{169}$

The exact dist'n of the number of plushies won?

$$\text{Binomial}\left(n = 169, p = \frac{1}{169}\right)$$

The exact probability is

$$1 - \mathbb{P}[\text{no plushies won}] = 1 - \left(\frac{168}{169}\right)^{169}$$

The approximate dist'n of the number of plushies won?

Criterion: $n \cdot p = 169 \cdot \frac{1}{169} = 1$ $\Rightarrow$ Poisson approximation

$$1 - \mathbb{P}[\text{no plushies won}] \cong 1 - e^{-\mu} \cdot \frac{\mu^0}{\underbrace{0!}_{=1}} = 1 - e^{-\mu}$$

w/ $\mu = np = 1$

Approximately $1 - e^{-1}$ $\square$

Problem. A 13-sided die is rolled 169 times.
Every time a number equal to 5 or more is obtained, we get a candy bar.
What is the probab. we get @ least 20 candy bars?

$\longrightarrow$: Dist'n of the number of candy bars won:

$$\text{Binomial}\left(n=169, \tfrac{9}{13}\right).$$

The exact probab. is:

$$\sum_{k=20}^{169} \binom{169}{k} \left(\frac{9}{13}\right)^k \left(\frac{4}{13}\right)^{169-k}$$

The approximate probability of the event:

Check: $n \cdot p = 169 \cdot \frac{9}{13} = 13 \cdot 9 = 117 > 10$

$n \cdot (1-p) = 169 \cdot \frac{4}{13} = 13 \cdot 4 = 52 > 10$ ✓

We use the normal approximation.

$$\mu = n \cdot p = 117$$

$$\sigma = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{\cancel{169} \cdot \frac{9}{\cancel{13}} \cdot \frac{4}{\cancel{13}}} = 6$$

$\mathbb{P}$[a least 20 candy bars won] $\approx$

$$\approx 1 - \Phi\left(\frac{20 - \frac{1}{2} - 117}{6}\right) = 1 - \Phi\left(-\frac{97.5}{6}\right) =$$

$$= 1 - \Phi(-16.25) \approx 1$$

□