M362K: January 26th, 2023.

Terminology:

probability = distribution

probability = likelihood = chance = long-term frequency

Problem. Two cards are drawn from an ordinary deck @ random. Find the probability that one card is a spade and one is a heart.

$\longrightarrow$: @ random $\longleftrightarrow$ all outcomes are equally likely

Method I. $\boxed{\text{ORDERED PAIRS}}$

$\Omega = \{(c_1, c_2)\}$

Ω... the set of all ordered pairs of cards (w/out replacement)

$\#(\Omega) = \underline{52 \cdot 51}$

$A = \{$one card is a spade and the other is a heart$\}$

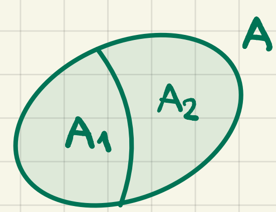
$A_1 = \{1^{st}$ card a spade, 2^{nd} a heart$\}$

$A_2 = \{1^{st}$ card a heart, 2^{nd} a spade$\}$

Q: Is this really a partition of A?

(i) $A_1 \cup A_2 \overset{?}{=} A$ ✓

(ii) $A_1 \cap A_2 = \emptyset$



By def'n:

$$\mathbb{P}[A] = \frac{\#(A)}{\#(\Omega)}$$

$$\mathbb{P}[A] = \frac{\#(A_1) + \#(A_2)}{\#(\Omega)} = \mathbb{P}[A_1] + \mathbb{P}[A_2]$$

In this problem: $\#(A_1) = \underline{13 \cdot 13} = 13^2 = \#(A_2)$

$$\Rightarrow \quad \mathbb{P}[A] = \frac{\cancel{2} \cdot 13^{2}}{\cancel{52} \cdot 51} = \boxed{\frac{13}{102}}$$

$\square$

Method II. $\boxed{\text{Subsets of the deck w/ two elements each}}$

$\tilde{\Omega} = \{ \{c_1, c_2\} \dots \}$

$\tilde{\Omega}$ ... the set of all subsets w/ two elements (cards) in an ordinary deck

$$\#(\tilde{\Omega}) = \boxed{\frac{52 \cdot 51}{2}} = \boxed{\binom{52}{2}}$$

In general, the number of subsets of k elements out of a set w/ n elements $(0 \leq k \leq n)$ is denoted by

$$\boxed{\binom{n}{k}} \quad (\text{"n choose k"}).$$

Q: How is this calculated?

→ • With j elements, in how many ways can I order them?

$$j! := \underline{j} \cdot \underline{(j-1)} \cdot \underline{(j-2)} \cdot \dots \cdot \underline{1} \qquad (j \text{ factorial})$$

• With n elements available, in how may ways can I fill k slots?

$$\underset{\#1}{\underline{n}} \cdot \underset{\#2}{\underline{(n-1)}} \cdot \underset{\#3}{\underline{(n-2)}} \cdot \underset{\dots}{\underline{\dots}} \cdot \underset{\#k}{\underline{(n-k+1)}}$$

• Every combination above has $\underline{k!}$ rearrangements (permutations).

• Since we don't care about the order in the subsets, the total # of subsets will be:

$$\boxed{\binom{n}{k} = \frac{n \cdot (n-1)(n-2) \cdots (n-k+1)}{k!}}$$

Note:

$$\binom{n}{k} = \boxed{\frac{n\cdot(n-1)(n-2)\cdots(n-k+1)}{k!}} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{k!\,(n-k)!}$$

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

binomial coefficients

$\tilde{A} = \{$ one heart & one spade $\}$

$\#(\tilde{A}) = 13\cdot 13 = 13^2$

$$\mathbb{P}[\tilde{A}] = \frac{\#(\tilde{A})}{\#(\tilde{\Omega})} = \frac{13^{\cancel{2}}}{\dfrac{{}^{2}\cancel{{}^{4}}\cancel{52}\cdot 51}{\cancel{2}}} = \boxed{\frac{13}{102}}$$

$\square$