M362K: January 24th, 2024.

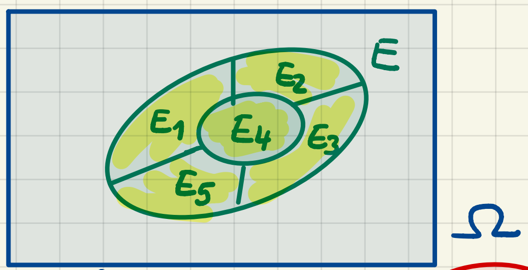
Section 1.3. $\boxed{\text{Distributions.}}$

Def'n. We say that an event E is partitioned into events $E_1, E_2, \dots, E_n$ if:

(i) $E = E_1 \cup E_2 \cup \cdots \cup E_n$

and

(ii) $E_1, E_2, \dots, E_n$ are mutually exclusive, i.e.,



$E_j \cap E_k = \emptyset$ for all $j, k \in \{1, \dots, n\}$ and $j \neq k$

Rules of Proportion and Probability.

Let $\mathbb{P}$ be a function on subsets of Ω satisfying:

(i) non-negativity: For every event E: $\boxed{\mathbb{P}[E] \geq 0}$

(ii) additivity: For every event E and its partition $E_1, E_2, \dots, E_n$, we have

$$\boxed{\mathbb{P}[E_1] + \mathbb{P}[E_2] + \cdots + \mathbb{P}[E_n] = \mathbb{P}[E]}$$

(think: AREA or MASS)

(iii) "total one": $\boxed{\mathbb{P}[\Omega] = 1}$

Then, $\mathbb{P}$ is called a distribution or probability over Ω.

Example. Two fair coins are tossed.

Let

$E = \{\text{both results are the same}\}$

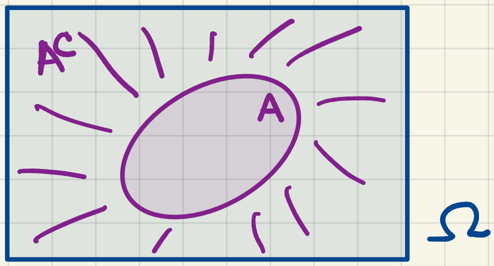
$$\boxed{\begin{array}{l} E_1 = \{HH\} \\ E_2 = \{TT\} \end{array}}$$ form a partition of E

$$\frac{2}{4} = \mathbb{P}[E] = \mathbb{P}[E_1] + \mathbb{P}[E_2] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

□

Rules.

The Complement Rule.



$$\boxed{\mathbb{P}[A^c] = 1 - \mathbb{P}[A]}$$

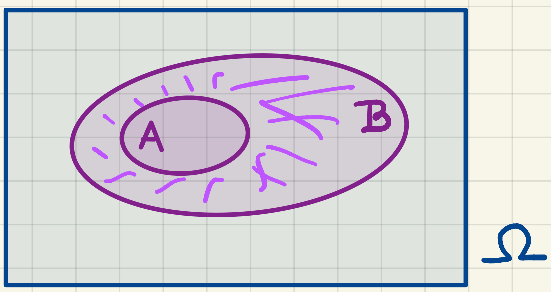
Pf.

A and A^c partition Ω

$$\mathbb{P}[A] + \mathbb{P}[A^c] = \mathbb{P}[\Omega] = 1$$

$\square$

The Difference Rule.



"if A, then B"

$\mathbb{P}[A] \leq \mathbb{P}[B]$

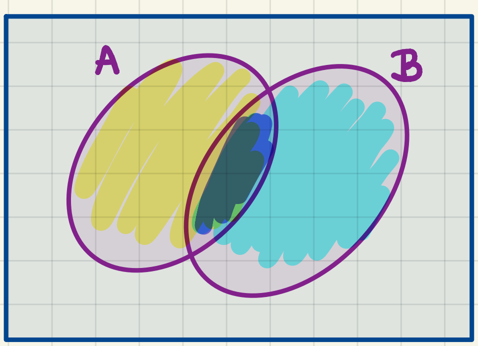
$\mathbb{P}[B \text{ but not } A] = \boxed{\mathbb{P}[B] - \mathbb{P}[A]}$

Pf. A and $\boxed{B \cap A^c}$ partition B

$$\mathbb{P}[B \cap A^c] + \mathbb{P}[A] = \mathbb{P}[B]$$

$\square$

Inclusion-Exclusion.



$$\mathbb{P}[A \cup B] = \cancel{X} = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$$

Problem. You are given $\mathbb{P}[A \cup B] = 0.7$ ☆
and $\mathbb{P}[A \cup B^c] = 0.9$ ☆☆

Find $\mathbb{P}[A]$.

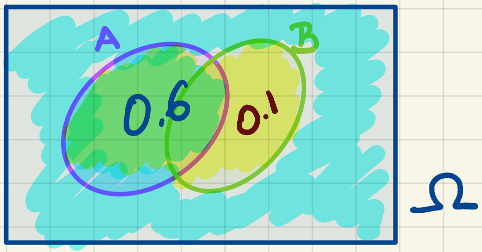
Method I: By the inclusion-exclusion formula, we have

$$\left.\begin{aligned} \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] &= 0.7 \\ \mathbb{P}[A] + \mathbb{P}[B^c] - \mathbb{P}[A \cap B^c] &= 0.9 \end{aligned}\right\} +$$

$$2\mathbb{P}[A] + 1 - \underbrace{\left(\mathbb{P}[A\cap B] + \mathbb{P}[A\cap B^c]\right)}_{\mathbb{P}[A]} = 1.6$$

$$\boxed{\mathbb{P}[A] = 1.6 - 1 = 0.6}$$

Method II.



$\square$

Problem. The first urn contains 10 balls: 4 red and 6 blue. The second urn contains 16 red balls and an unknown number of blue balls.

A single ball is drawn from each urn.

The probability that both balls are the same color is 0.44.

Find the number of blue balls in the second urn.

$\longrightarrow$:

$$\mathbb{P}[\text{same color}] = \mathbb{P}[\text{both red}] + \mathbb{P}[\text{both blue}]$$

x... the number of blue balls in second urn

$$0.44 = \frac{4}{10}\cdot\frac{16}{16+x} + \frac{6}{10}\cdot\frac{x}{16+x} \quad \Big/ \cdot 10(16+x)$$

$$4.4(16+x) = 4\cdot 16 + 6\cdot x$$

$$4.4\cdot 16 + 4.4\cdot x = 4\cdot 16 + 6\cdot x$$

$$1.6x = 0.4\cdot 16 \qquad \boxed{x = 4}$$

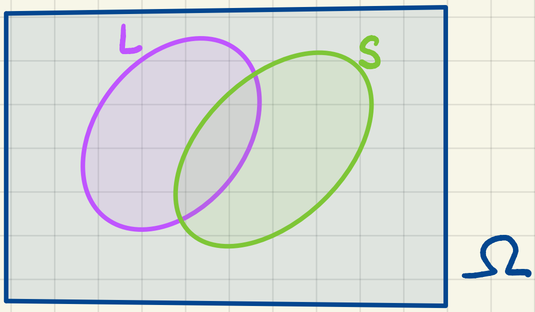
$\square$

Problem. The probability that a visit to a PCP's office results in neither lab work nor referral to a specialist is 35%.

Of those coming to the PCP's office, 30% are referred to a specialist and 40% require lab work.

Find the probability that a visit to a PCP's office results in both lab work and a specialist referral.

→ :



Inclusion · Exclusion.

$$\underbrace{\mathbb{P}[L \cup S]} = \mathbb{P}[L] + \mathbb{P}[S] - \boxed{\mathbb{P}[L \cap S]}$$

$$0.65 = 0.4 + 0.3 - \mathbb{P}[L \cap S]$$

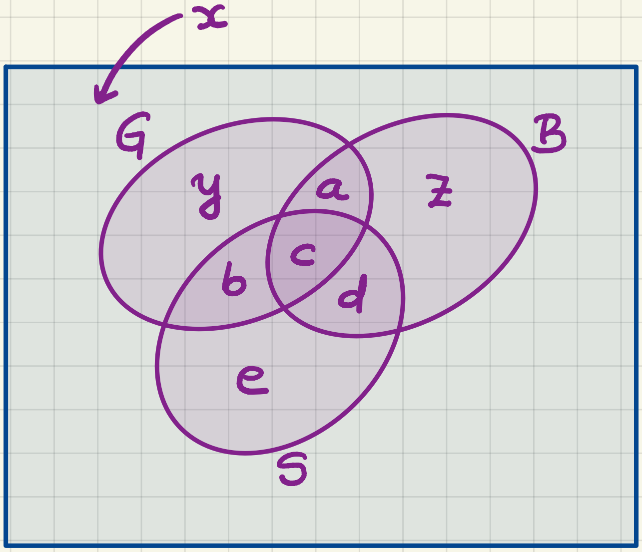
$$\Rightarrow \boxed{\mathbb{P}[L \cap S] = 0.05}$$

□

Problem. A survey of a group's viewing habits over the last year revealed:

(i) 28% watched gymnastics
(ii) 29% –\|\|– baseball
(iii) 19% –\|\|– soccer
(iv) 14% –\|\|– gymnastics & baseball
(v) 12% –\|\|– baseball & soccer
(vi) 10% –\|\|– gymnastics & soccer
(vii) 8% –\|\|– all three sports

Calculate the percentage of the group that watched none of the three sports in the last year.



$a+b+c+y=0.28$ $\quad y=0.12$ ✓

$a+c+d+z=0.29$ $\quad z=0.11$ ✓

$b+c+d+e=0.19$ $\quad e=0.05$ ✓

$a+c=0.14$ $\quad a=0.06$ ✓

$c+d=0.12$ $\quad d=0.04$ ✓

$b+c=0.10$ $\quad b=0.02$ ✓

$c=0.08$

$a+b+c+d+e+y+z=0.48$

$\Rightarrow \quad x=0.52$