M362K: February 26^{th}, 2024.

Normal Distribution.

Start w/ the standard normal density:

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}}$$

Take a linear transform: $\boxed{x = \mu + \sigma \cdot z}$ $\mu \in \mathbb{R}, \sigma > 0$

μ: center (mean)

σ: spread (standard deviation)

After the transform, we get the normal density w/ parameters μ and σ:

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for all } x \in \mathbb{R}}$$

Necessary to preserve area 1 under the curve!

The "inverse" linear transform $\boxed{x \longmapsto z = \frac{x-\mu}{\sigma}}$

This is expressing your raw scores in standard units.

Problem. Assume the students' scores are normal w/ mean 74 and standard deviation 12.

(a) Find the raw scores corresponding to these standard scores:

(i) -1 $\longrightarrow$: $x = \mu + z \cdot \sigma = 74 + (-1) \cdot 12 = 62$

(ii) 0.5 $\longrightarrow$: $x = 74 + 0.5 \cdot 12 = 80$

(iii) 1.25 $\longrightarrow$: $x = 74 + 1.25 \cdot 12 = 89$

(b) Find the scores in standard units for students w/ the following raw scores:

(i) 65 $\longrightarrow$: $z = \frac{x-\mu}{\sigma} = \frac{65-74}{12} = -0.75$

(ii) 74 $\longrightarrow$: 0

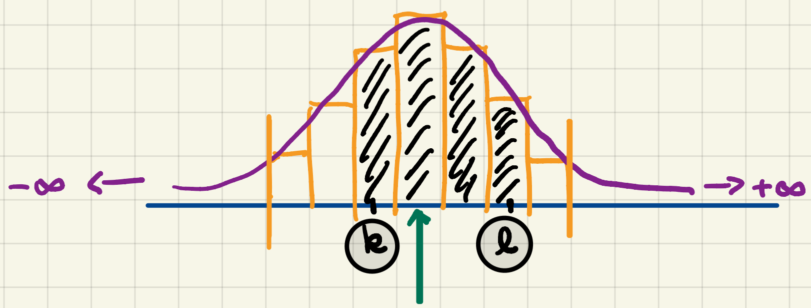
(iii) 86 $\longrightarrow$: $z = 1$

Usage: The probability that a normal distribution w/ parameters μ and σ assigns to the interval $[c,d]$ is

$$\boxed{\Phi\left(\frac{d-\mu}{\sigma}\right) - \Phi\left(\frac{c-\mu}{\sigma}\right)} \checkmark$$

The Normal Approximation to the Binomial Distribution

(The deMoivre-Laplace Theorem).



Goal: We want to approximate the probability of the $\boxed{\text{EVENT}}$ of the form:

$$\{\text{from } \underbrace{k \text{ to } \ell}_{\text{integers}} \text{ successes in } n \text{ trials}\}$$

We have reduced the problem to that of the choice of parameters μ and σ of the bell curve.

$\boxed{\text{SET:}}$

- $\mu = n \cdot p$ ("clear" since we want to center the bell curve @ the center of mass of the binomial histogram)
- $\sigma = \sqrt{n \cdot p \cdot (1-p)}$ (Have Faith!!! Or look @ Section 2.3. in Pitman)

Next:
- center the histogram, i.e., subtract the mean $\mu = n \cdot p$
- rescale the histogram, i.e., divide the difference by the std deviation $\sigma = \sqrt{np(1-p)}$

This linear transform brings the "raw" # of successes to STANDARD UNITS.

We hope $\boxed{\Phi\left(\frac{\ell-\mu}{\sigma}\right)-\Phi\left(\frac{k-\mu}{\sigma}\right)}$ will be a good approximation?

Example. Consider 100 tosses of a fair coin.

$\boxed{\text{Binomial}(n=100,\ p=0.5)}$

Q: What is the probability of getting exactly 50 Heads?

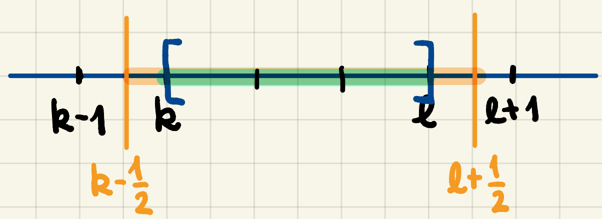
$\longrightarrow$: exact: $\binom{100}{50}\frac{1}{2^{100}} > 0$

our approximation: $\mu = n\cdot p = 100\left(\frac{1}{2}\right)=50$

$\sigma = \sqrt{n\cdot p\cdot(1-p)} = 5$

$$\Phi\left(\frac{50-50}{5}\right)-\Phi\left(\frac{50-50}{5}\right)=0$$

¡The same would happen for any $k=\ell$!



We resort to the continuity correction:

integers

$\mathbb{P}[\{\text{from } k \text{ to } \ell \text{ successes in } n \text{ trials}\}] =$

$= \mathbb{P}[\{\text{from } k-\frac{1}{2} \text{ to } \ell+\frac{1}{2} \text{ successes in } n \text{ trials}\}]$

$$\approx \boxed{\Phi\left(\frac{\ell+\frac{1}{2}-\mu}{\sigma}\right)-\Phi\left(\frac{k-\frac{1}{2}-\mu}{\sigma}\right)}$$

Special case: $k=\ell$

$$\Phi\left(\frac{k-\mu}{\sigma}+\frac{1}{2\sigma}\right)-\Phi\left(\frac{k-\mu}{\sigma}-\frac{1}{2\sigma}\right)>0$$