

M362K: February 28th, 2024.

Standard Normal Distribution.

The standard normal density function is

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}$$

The standard normal cumulative distribution function is

$$\Phi(z) = \int_{-\infty}^z \varphi(u) du \quad \text{for all } z \in \mathbb{R}$$

Usage: Under the std normal distribution, the probability of the event $(a, b]$ will be

$$\mathbb{P}[(a, b]] = \Phi(b) - \Phi(a)$$

Note: There is no "simple exact formula" for Φ !

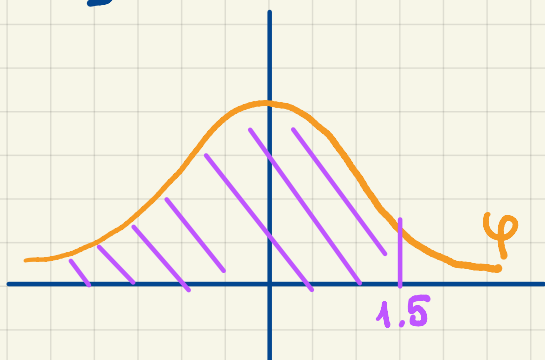
But, $\int$ we have standard normal tables (Appendix 5).
 $\bullet$ we also have `pnorm` in R

Problem. Consider the standard normal distribution.

Q: What is the probability of the event $(-\infty, 1.5]$?

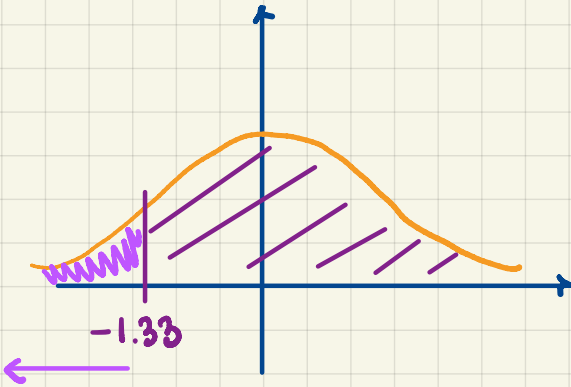
→: Using the std normal tables: $\Phi(1.5) = 0.9332$

Using R: `pnorm(1.5) = 0.93312928`



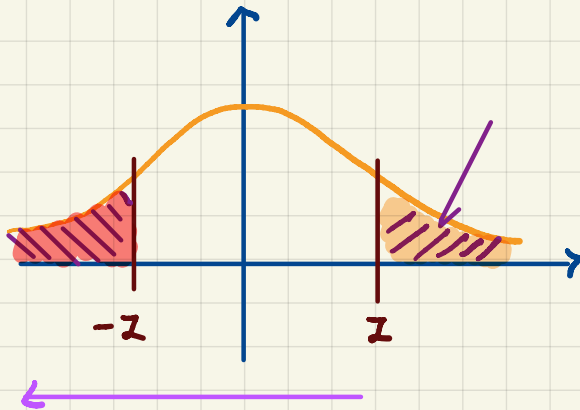
Q: What is the probability of the event $(-1.33, +\infty)$?

→:



$$1 - \Phi(-1.33) = 1 - 0.0918 = 0.9082$$

$$1 - \text{pnorm}(-1.33) = 0.9082409$$



$$1 - \Phi(z) = \Phi(-z) \text{ for all } z$$

We could have done:

$$\Phi(1.33) = 0.9082$$

Q: What is the probability of the event $[-2.56, 1.73]$?

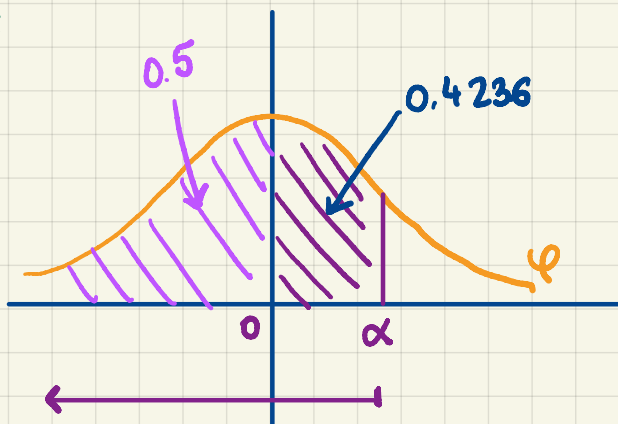
$$\Phi(1.73) - \Phi(-2.56) = 0.9582 - 0.0052 = 0.9530$$

$$\text{pnorm}(1.73) - \text{pnorm}(-2.56) = 0.9529513$$



Problem. Assume that the number α is such that the standard normal distribution assigns the probability of 0.4236 to the interval $[0, \alpha]$. Find α .

→:



$$0.5 + 0.4236 = \Phi(\alpha)$$

$$\Phi(\alpha) = 0.9236$$

$$\alpha = \Phi^{-1}(0.9236)$$

$$\Phi^{-1}(0.9236) = 1.43$$

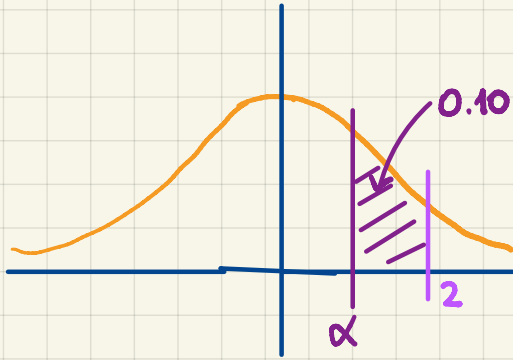
Usually, the inverse of the cdf is called the **quantile function**.

$$\text{qnorm}(0.9236) = 1.429711$$



Problem. Find α such that the area underneath ϕ over the interval $[\alpha, 2]$ equals 10%.

→:



$$\Phi(2) - \Phi(\alpha) = 0.1$$

$$\Phi(\alpha) = \Phi(2) - 0.1$$

$$= 0.9772 - 0.1 = 0.8772$$

$$\alpha = \Phi^{-1}(0.8772) \approx 1.16$$

Find the closest probability
IN THE TABLES.
In this case, it was 0.8770.

$$\text{qnorm}(\text{pnorm}(2) - 0.1) = 1.161348$$



Normal Distributions.

In general, we can change the center of the bell curve and its spread by introducing parameters:

and μ ... mean (measure of center)

σ ... standard deviation (measure of spread)

x raw units $\longrightarrow$ z standard units

$$x = \mu + \sigma \cdot z$$

$$z = \frac{x - \mu}{\sigma}$$

linear transforms