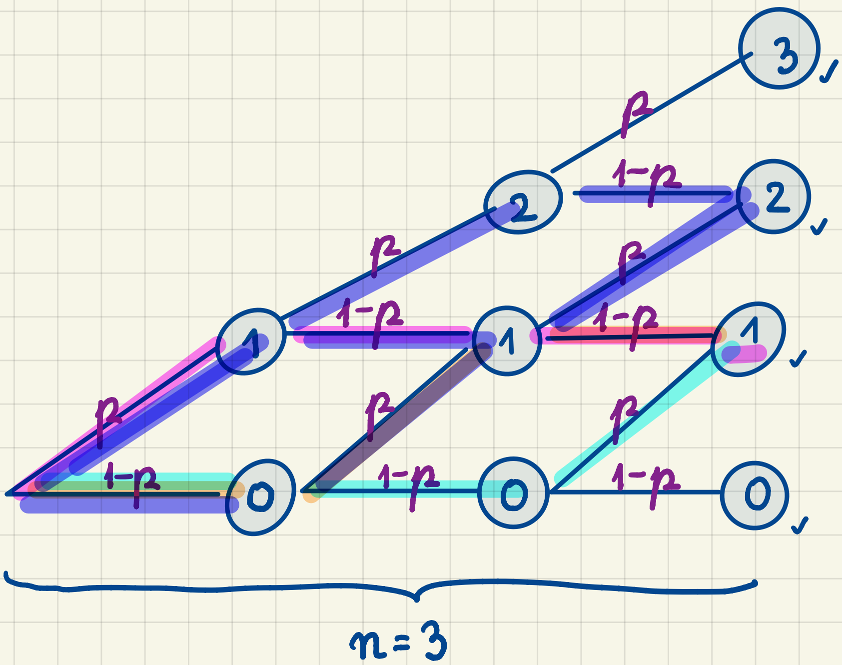
M362K: February 19th, 2024.

Binomial Distribution.

n independent, identically distributed Bernoulli trials.

p ... probability of success in every trial.



$p_3 = p^3$

$p_2 = 3 \cdot p^2(1-p)$

$p_1 = 3\, p(1-p)^2$

$p_0 = (1-p)^3$

In general: $\boxed{\Omega = \{0, 1, \dots, n\}}$

$k \in \Omega$: $\quad p_k := \mathbb{P}[\{k\}] =$

$= \mathbb{P}[k \text{ successes in } n \text{ trials}] = \binom{n}{k} \underline{p^k (1-p)^{n-k}}$

probab. of every path w/ exactly k successes

# of paths w/ exactly k successes

Note: $\quad 1 \overset{?}{=} \sum_{k=0}^{n} p_k = \sum_{k=0}^{n} \binom{n}{k} \underset{x}{p^k} (\underbrace{1-p}_{y})^{n-k} =$ $\boxed{\text{(binomial theorem)}}$

$= (\cancel{p} + (1-\cancel{p}))^n = 1^n = 1$

Binomial Thm:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}$$

Problem. Consider four independent tosses of a fair coin.
Is it more likely to have 2 Heads and 2 Tails
or a different number of Heads and Tails?

$\longrightarrow$: n... # of cointosses, i.e., $n=4$

p... probab. of Heads in a single toss, i.e., $p=\frac{1}{2}$

binomial dist'n : Binomial$(n=4, p=\frac{1}{2})$

or $b(n=4, p=\frac{1}{2})$

$$\mathbb{P}[\text{exactly } 2\text{Hs and } 2\text{Ts}] = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{4-2}$$

$$= \frac{\cancel{4}^{2}\cdot 3}{\cancel{2}} \cdot \frac{1}{2^{\cancel{4}3}} = \frac{3}{8}$$

$$\mathbb{P}[\text{different \# of Hs and Ts}] = \frac{5}{8}$$ ✓

□

Problem. A man fires 8 shots @ a target.
Assume that his shots are independent, and
that each shot hits bull's eye w/ probab. 0.7.

Q: What is our model for the number of times bull's eye is hit?

$\longrightarrow$: Binomial$(n=8, p=0.7)$.

Q: What is the probability that he hits bull's eye exactly 4 times?

$\longrightarrow$: $\mathbb{P}[\text{exactly 4 successes in 8 trials}] =$

$$= \binom{8}{4}(0.7)^4(1-0.7)^4$$

$$= \frac{\cancel{8}^{2}\cdot 7\cdot \cancel{6}\cdot 5}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1}(0.7)^4(0.3)^4 = \boxed{70(0.21)^4} = 0.1361367$$

Q: Given that he hit the bull's eye @ least twice, what is the chance that he hit bull's eye exactly 4 times?

→:

$$\mathbb{P}[\text{exactly 4 hits in 8 shots} \mid \text{@ least 2hits in 8 shots}] = (\text{def'n})$$

$$= \frac{\mathbb{P}[\text{exactly 4 hits in 8 shots and at least 2hits in 8 shots}]}{\mathbb{P}[\text{@ least 2 hits in 8 shots}]}$$

$$= \frac{\mathbb{P}[\text{exactly 4 hits in 8 shots}]}{1-\mathbb{P}[\text{no hits}]-\mathbb{P}[\text{exactly 1 hit}]} =$$

$$= \frac{70(0.21)^4}{1-(0.3)^8 - 8\cdot 0.7\cdot(0.3)^7} = 0.1363126$$

Q: Given that the first two shots hit bull's eye, what is the chance of exactly 4 hits in the 8 shots?

→:

$$\overbrace{\underset{1.}{\underline{S}}\quad \underset{2.}{\underline{S}}}^{\text{two successes}}\ \Big|\ \overbrace{\underset{3.}{\underline{\quad}}\quad \underset{4.}{\underline{\quad}}\quad \underset{5.}{\underline{\quad}}\quad \underset{6}{\underline{\quad}}\quad \underset{7.}{\underline{\quad}}\quad \underset{8.}{\underline{\quad}}}^{\text{exactly 2 hits in these 6 trials}}$$

Answer = $\mathbb{P}[\text{exactly 2 hits in the 6 trials}]$ =

$$= \binom{6}{2}(0.7)^2(0.3)^{6-2} = \frac{\cancel{6}^{\,3}\cdot 5}{\cancel{2}}\cdot(0.7)^2\cdot(0.3)^4$$

$$= 15\cdot(0.063)^2 = 0.059535$$

$\square$