M362K: February 16th, 2024.

Geometric Distribution [revisited].

Start w/ a sequence of independent, identically distributed Bernoulli trials.

the same probability p of success in a single trial.

Repeat the trials until the first success.
Count how many trials it took (including the success).

Outcome Space: $\boxed{\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}}$

Distribution: $\boxed{p_k := \mathbb{P}[\{k\}] = (1-p)^{k-1} \cdot p \quad \text{for all } k \in \mathbb{N}}$

Example. Consider a house in a 100-yr flood plane, then this can be interpreted as $\boxed{p = \frac{1}{100}}$ being the chance of a flood in a single year.

Assume independent years.

$$\mathbb{P}[\text{no flood in next 30 years}] = \left(\frac{99}{100}\right)^{30} = 0.7397$$

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Problem Set # 2

Sequences of events.

Problem 2.1. Roger is playing darts. His throws are all mutually independent, and he has a probability 0.3 of hitting the bull's eye in any single throw. How many darts should Roger throw so that there is at least an 80% probability of hitting bull's eye at least once?

$\rightarrow$:

$\mathbb{P}[\text{no bulls eye in } n \text{ throws}] = (0.7)^n$

$1-(0.7)^n \geq 0.8$

$(0.7)^n \leq 0.2$

ln |

$n \cdot \ln(0.7) \leq \ln(0.2) \quad /: \ln(0.7) < 0$

$n \geq \frac{\ln(0.2)}{\ln(0.7)} = 4.51 \quad \Rightarrow \quad \boxed{n \geq 5}$

$\square$

Problem 2.2. Audrey and Evie take turns tossing a fair coin with Audrey having the first turn. Whoever gets *Heads* first wins the game.

(i) What's the probability that Evie wins the game?
(ii) Is it possible to weight the coin so that the game is fair, i.e., with what probability p should *Heads* appear so that Audrey and Evie are equally likely to win the game?

p_E ... probab. that Evie wins

Method I.

$$p_E = \mathbb{P}[\text{it takes an even \# of trials}]$$
$$= p_2 + p_4 + \dots + p_{2k} + \dots$$
$$= \underbrace{(1-p)\cdot p}_{a} + (1-p)^3 \cdot p + \dots + (1-p)^{2k-1} \cdot p + \dots$$
$$= (1-p)\cdot p\,(1 + \overset{r}{(1-p)^2} + (1-p)^4 + \dots)$$
$$= \frac{(1-p)p}{1-(1-p)^2} = \frac{(1-p)\cancel{p}}{(\cancel{1-1+p})(1+1-p)} = \boxed{\frac{1-p}{2-p}}$$

$\boxed{\text{For } p = \frac{1}{2}:}$ $\boxed{p_E = \frac{1}{3}}$

(ii) Try to solve:

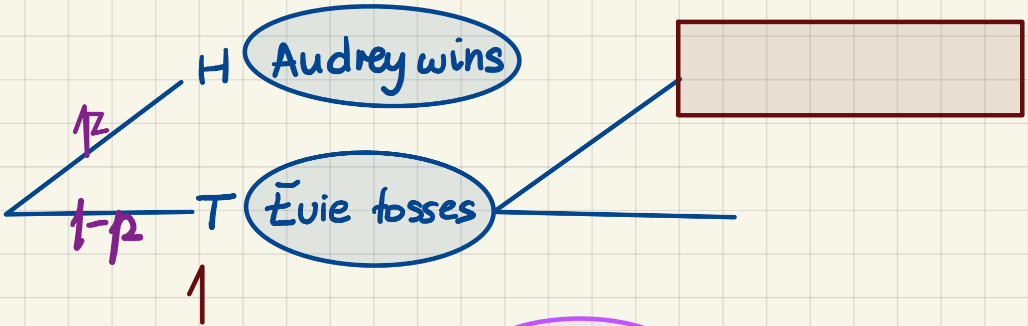
$$\frac{1-p}{2-p} = \frac{1}{2}$$
$$\cancel{2} - 2p = \cancel{2} - p$$

$\boxed{p = 0 \text{ means they play forever}}$

Method II.

p_E … probab. that Evie wins

p_A … probab. that Audrey wins



$$p_E = (1-p)\,p_A$$

$$p_E + p_A = 1$$

$$(1-p)\,p_A + p_A = 1$$

$$(2-p)\,p_A = 1$$

$$p_A = \frac{1}{2-p} \quad \Rightarrow \quad \boxed{p_E = \frac{1-p}{2-p}}$$

$\square$

Problem 2.3. *Source: Problem #1.5.6 from Pitman.*
Suppose you roll a fair six-sided die repeatedly until the first time you roll a number that you have rolled before. Let p_r denote the probability that you roll the die exactly r times.

(i) Without calculation, write down the value of $p_1 + p_2 + ... + p_{10}$. **Explain.** 1

(ii) For each $r = 1, 2, \ldots$ calculate the probability p_r. ✓

(iii) Verify arithmetically that your response to (i) was correct.

$p_1 = 0$

$p_2 = \frac{1}{6}$

$p_3 = \frac{5}{6} \cdot \frac{2}{6}$

$p_4 = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6}$

$p_5 = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{4}{6}$

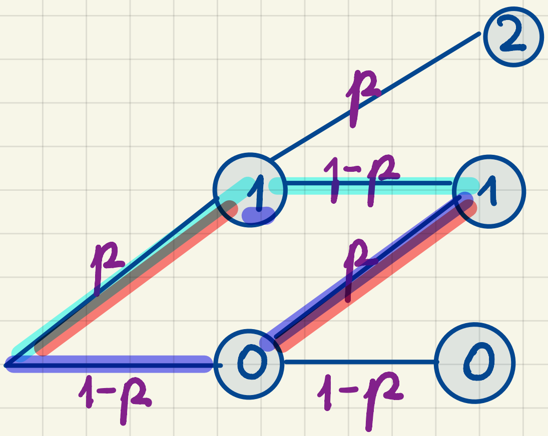
$p_6 = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{5}{6}$

$p_7 = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} \cdot 1$

$\boxed{p_r = 0 \text{ for } r \geq 8}$

Section 2.1.

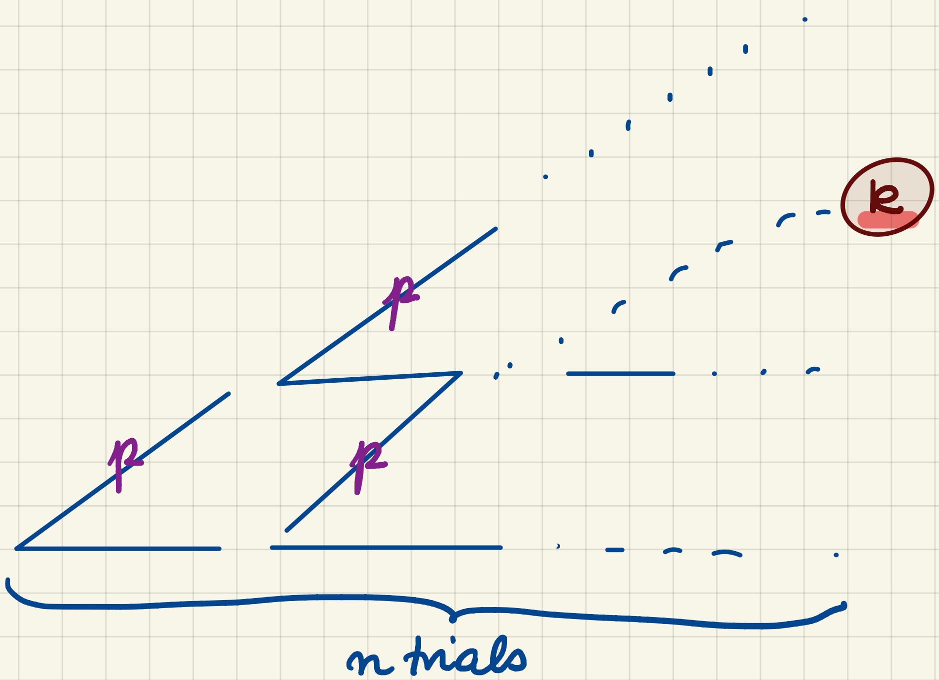
Binomial Distribution.



w/ probab. p^2

w/ probab. $2p(1-p)$

w/ probab. $(1-p)^2$



w/ probab. $\binom{n}{k}p^k(1-p)^{n-k}$

Outcome Space: $\Omega = \{0, 1, 2, \dots, n-1, n\}$

Distribution:

$$\boxed{p_k = \binom{n}{k} p^k (1-p)^{n-k} \qquad k \in \Omega}$$

We write: Binomial(n, p) or $\boxed{b(n,p)}$