M362K: February 12th, 2024.

Section 1.6.

Review: **Def'n.** Two events E and F are independent if

$$\boxed{\mathbb{P}[E \cap F] = \mathbb{P}[E] \cdot \mathbb{P}[F]}$$

Def'n. Let $A, B,$ and C be three events on the same Ω.
If

$$\boxed{\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \cdot \mathbb{P}[B] \cdot \mathbb{P}[C]}$$

and
analogous equalities hold for any of the events substituted by their complements,
then, we say that $A, B,$ and C are independent.

Note. There are $2^3 = 8$ equlities total that must hold.

Def'n. We say that $A_1, A_2, \dots, A_n$ are independent if the analogous identity

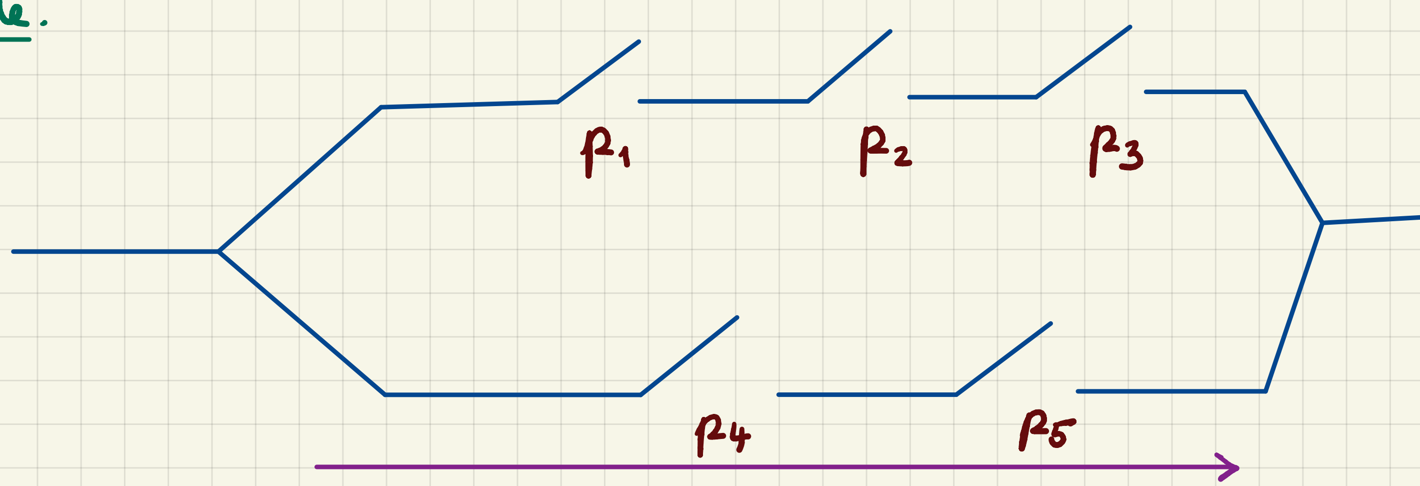
$$\mathbb{P}[A_1 \cap \cdots A_n] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2] \cdots \mathbb{P}[A_n]$$

and all the other equalities obtained by complementation are true.

Note. The total number of equalities is 2^n.

Easier rule:
Say that $A_1, A_2, \dots, A_n$ are independent.
Then, any event determined by a subcollection of these events is independent of any other event determined by a subcollection of remaining events.

Example.



Assume: Switches are independent.

$\mathbb{P}$[current flows through the system] =

$= \mathbb{P}$[@ least one of "top" and "bottom" is working]

$= \mathbb{P}[T \cup B]$

$= \mathbb{P}[T] + \mathbb{P}[B] - \mathbb{P}[T \cap B]$

$= p_1 \cdot p_2 \cdot p_3 + p_4 \cdot p_5 - p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5$ $\square$

Def'n. We say that $A_1, A_2, \ldots, A_n$ are pairwise independent if A_i and A_j are independent for all $i \neq j$.

Note. This is $\boxed{\text{weaker}}$ than the notion of independence.

Example. Toss a fair coin twice independently.

$H_1 = \{1^{st} \text{ toss is heads}\} = \{HH, HT\}$ $\quad \mathbb{P}[H_1] = \frac{1}{2}$

$H_2 = \{2^{nd} \text{ toss is heads}\} = \{HH, TH\}$ $\quad \mathbb{P}[H_2] = \frac{1}{2}$

$D = \{\text{results were different}\} = \{HT, TH\}$ $\quad \mathbb{P}[D] = \frac{1}{2}$

Claim. These events are pairwise independent.

$\rightarrow$: $\mathbb{P}[H_1 \cap H_2] = \mathbb{P}[HH] = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}[H_1] \cdot \mathbb{P}[H_2]$ ✓

$\mathbb{P}[H_1 \cap D] = \mathbb{P}[HT] = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}[H_1] \cdot \mathbb{P}[D]$ ✓

$\mathbb{P}[H_2 \cap D] = \mathbb{P}[TH] = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = \mathbb{P}[H_2] \cdot \mathbb{P}[D]$ ✓

However:

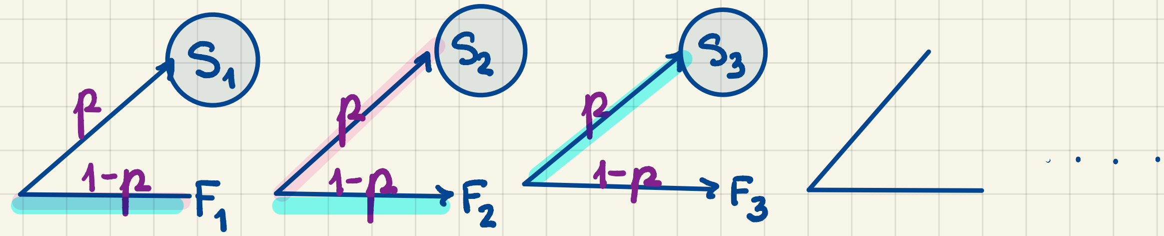
$\mathbb{P}[H_1 \cap H_2 \cap D] = 0 \neq \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ $\quad \boxed{\text{NOT INDEPENDENT}}$

$\square$

Sequences of Events.

Example. Geometric Distribution

We repeatedly, independently toss a coin such that the probab. of Heads is equal to p.
We toss the coin until the first time we get Heads, i.e., until the first "success", counting the total number of tosses (Bernoulli trials).



Q: What is the probab. that it takes three or fewer tosses?

→: $1 - \mathbb{P}[\text{failures in the first three tosses}] = \underline{1-(1-p)^3}$

same as

$$\mathbb{P}[S_1] + \mathbb{P}[S_2] + \mathbb{P}[S_3] = \underline{p + (1-p)\cdot p + (1-p)^2 \cdot p}$$
$$= p\left(1 + (1-p) + (1-p)^2\right)$$

The appropriate outcome space for this experiment:

$$\boxed{\Omega = \{1, 2, 3, \ldots\} = \mathbb{N}}$$

$$p_k := \mathbb{P}[\{k\}] = \mathbb{P}[\text{the } k^{th} \text{ trial is the first success}] \quad \text{for } k = 1, 2, \ldots$$

Then,

$p_1 = p,$

$p_2 = (1-p)p,$

$p_3 = (1-p)^2 p,$

$\vdots$

$p_k = (1-p)^{k-1} \cdot p,$

$\vdots$

Geometric Distribution w/ Parameter p

Def'n. Let Ω be an outcome space.
A function $\mathbb{P}$ on the set of "nice" subsets of Ω is a **probability distribution** (or **probability measure**) if:

(i) $\mathbb{P}[\emptyset] = 0$,

(ii) for all pairwise disjoint $\{A_1, A_2, \ldots\}$ events in Ω

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mathbb{P}[A_n]$$

(iii) $\mathbb{P}[\Omega] = 1$.