M362K: February 5^{th}, 2024.

Problem 1.3. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability that both are spades.

→: Method I.

$\#(\Omega_I)$ = total number of pairs $= \binom{52}{2}$

$\#(E)$ = total number of pairs of spades $= \binom{13}{2}$

$$\mathbb{P}[\text{both spades}] = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{\frac{\cancel{13}\cdot\cancel{12}^{3}}{\cancel{2!}}}{\frac{\cancel{52}^{\cancel{4}}\cdot\cancel{51}^{17}}{\cancel{2!}}} = \frac{1}{17}$$

Method II.

$A_i = \{\text{the } i^{th} \text{ card is a spade}\}$ $i=1,2$

$$\mathbb{P}[A_1\cap A_2] = \mathbb{P}[A_1]\cdot\mathbb{P}[A_2|A_1] = \frac{\cancel{13}^{1}}{\cancel{52}_{\cancel{4}_1}}\cdot\frac{\cancel{12}^{\cancel{3}}}{\cancel{51}_{17}} = \frac{1}{17}$$

Multiplication Rule

$\square$

Def'n. We say that two events E and F are independent if

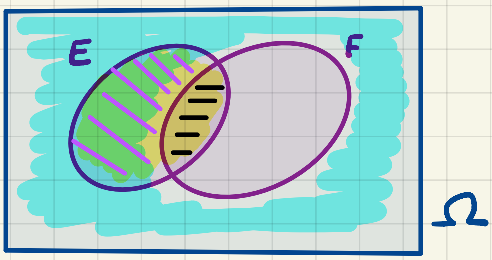
$$\boxed{\mathbb{P}[E\cap F]=\mathbb{P}[E]\cdot\mathbb{P}[F]}$$

Example. Assume that E and F are independent.

Claim. E and F^c are also independent. ✓

Proof. $\boxed{\text{Heuristics:}}$ $\mathbb{P}[E\,|\,F]=\mathbb{P}[E]$

$$\mathbb{P}[E\cap F^c] \overset{?}{=} \mathbb{P}[E]-\mathbb{P}[E\cap F] = \text{(independent } E\&F)$$



$$= \mathbb{P}[E]-\mathbb{P}[E]\cdot\mathbb{P}[F]$$
$$= \mathbb{P}[E]\left(1-\mathbb{P}[F]\right)$$
$$= \mathbb{P}[E]\cdot\mathbb{P}[F^c]$$

We also have: E^c and F are independent.
E^c and F^c are independent.

$\square$

Problem 1.4. If events E and F are independent, then they are necessarily mutually exclusive.

→ : $\boxed{\text{FALSE}}$

e.g., consider a roll of two dice

$$\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$$

w/ all outcomes equally likely

$$\mathbb{P}[(6,6)] = \frac{1}{36}$$

$$\mathbb{P}[\{6 \text{ on first die}\}] = \frac{1}{6} = \mathbb{P}[\{6 \text{ on } 2^{nd} \text{ die}\}]$$

not mutually exclusive,
but they are $\boxed{\text{independent}}$

□

Problem 1.5. The four standard blood types are distributed in a populations as follows:

$$A - 42\%$$
$$O - 33\%$$
$$B - 18\%$$
$$AB - 7\%$$

Assuming that people choose their mates independently of their blood type, find the probability that a randomly chosen couple from this population has the same blood type.

→:

$$\mathbb{P}[\text{same blood type}] =$$
$$= \mathbb{P}[\text{ both A or both O or both B or both AB}]$$
$$= \mathbb{P}[\text{both A}] + \mathbb{P}[\text{both O}] + \mathbb{P}[\text{both B}] + \mathbb{P}[\text{both AB}]$$
$$= (0.42)^2 + (0.33)^2 + (0.18)^2 + (0.07)^2 = \underline{0.3226}$$

$\square$

Problem 1.6. *Source: Sample P exam problems.*
An actuary studying the insurance preferences of automobile owners makes the following conclusions:

✓(i) An automobile owner is twice as likely to purchase collision coverage as disability coverage.
✓(ii) The event that an automobile owner purchases collision coverage is independent of the event that he or she purchases disability coverage.
✓(iii) The probability that an automobile owner purchases both collision and disability coverages is 0.15.

Calculate the probability that an automobile owner purchases neither collision nor disability coverage.

→ : $C = \{\text{collision coverage is purchased}\}$ $\quad \mathbb{P}[C] = p_C$

$D = \{\text{disability coverage is purchased}\}$ $\quad \mathbb{P}[D] = p_D$

$$\left.\begin{aligned} &(i) \Rightarrow p_C = 2p_D \\ &(ii) \Rightarrow \mathbb{P}[C \cap D] = p_C \cdot p_D \underset{(iii)}{=} 0.15 \end{aligned}\right\} \quad \begin{aligned} 2p_D \cdot p_D &= 0.15 \\ p_D &= \sqrt{0.075} \end{aligned}$$

We need $\mathbb{P}[C^c \cap D^c] \underset{\text{independence}}{=} \mathbb{P}[C^c] \cdot \mathbb{P}[D^c] = (1-p_C)(1-p_D)$

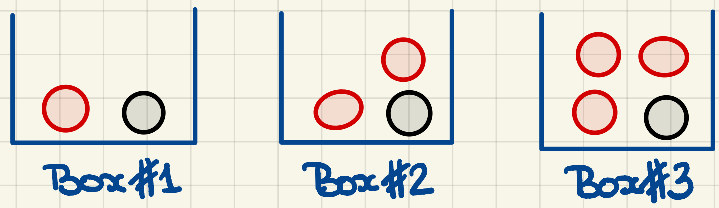
$$= (1-2\sqrt{0.075})(1-\sqrt{0.075})$$

$$= 1.15 - 3\sqrt{0.075} \quad \square$$

Section 1.5.

Bayes' Rule.

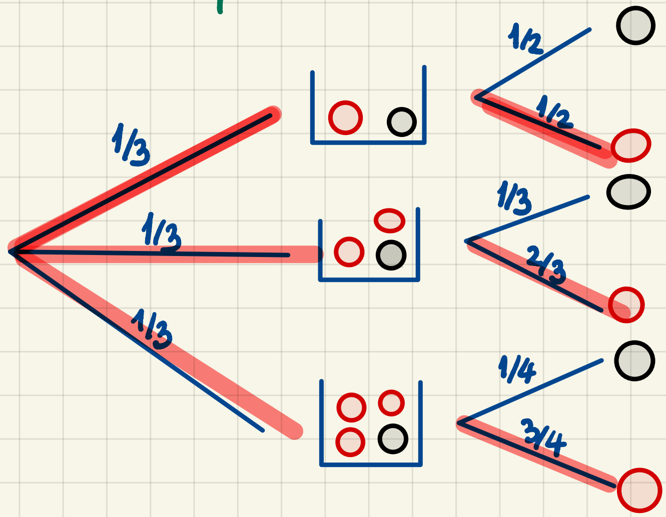
Example. Which Box?



Q: $\mathbb{P}[\text{Box } i \mid \text{Red}] = ?$ for all $i = 1, 2, 3$

$$\mathbb{P}[\text{Box } i \mid \text{Red}] = \frac{\boxed{\mathbb{P}[\text{Box } i \text{ and Red}]}}{\mathbb{P}[\text{Red}]} = \frac{\frac{1}{3} \cdot \frac{i}{1+i}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{3}{4}}$$

by def'n of conditional probab.



Theorem.

$$\mathbb{P}[B_i \mid A] = \frac{\mathbb{P}[B_i] \cdot \mathbb{P}[A \mid B_i]}{\mathbb{P}[B_1] \cdot \mathbb{P}[A \mid B_1] + \mathbb{P}[B_2] \cdot \mathbb{P}[A \mid B_2] + \cdots + \mathbb{P}[B_n] \cdot \mathbb{P}[A \mid B_n]}$$