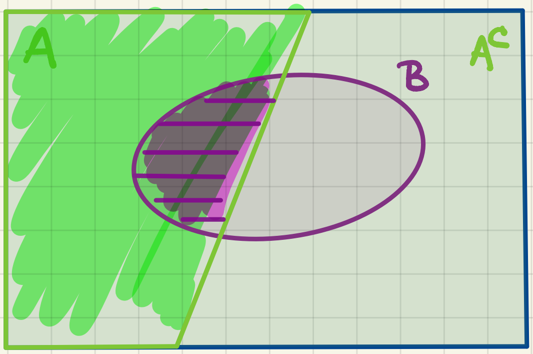
# M362K: February 2nd, 2024.

## Conditional Probability.




**Def'n.** Let A and B be two events.
Assume $\mathbb{P}[A]>0$.
We define the conditional probability of B given A as

$$\boxed{\mathbb{P}[B|A] = \frac{\mathbb{P}[\boxed{A\cap B}]}{\mathbb{P}[A]}}$$

We also say probability of B conditional on A.

Q: $\mathbb{P}[A|A] = 1$

Q: $\mathbb{P}[A^c|A] = 0$

**Example.** Consider a family w/ two pets.
One of the pets, a dog, enters the room.
Find the probability that the other pet is also a dog given that:

(i) nothing is known about the other pet;

(ii) you know the other pet was adopted first.

→: $\Omega = \{DD, DN, ND, NN\}$ all are equally likely

(i) the reduced outcome space:

$\Omega_i = \{DD, DN, ND\}$ cond. prob $= \frac{1}{3}$

(ii) the reduced outcome space:

$\Omega_{ii} = \{DD, ND\}$ cond. prob $= \frac{1}{2}$ □

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Problem Set # 1

Conditional probability. Independence.

**Problem 1.1.** Let $E$ and $F$ be any two events. If

$$\boxed{\mathbb{P}[E|F] > \mathbb{P}[E],}$$

then

$$\mathbb{P}[F|E] > \mathbb{P}[F].$$

True or false?

→ : We know that

$$\mathbb{P}[E|F] > \mathbb{P}[E]$$

By def'n

$$\frac{\mathbb{P}[E\cap F]}{\mathbb{P}[F]} > \mathbb{P}[E] \qquad / : \mathbb{P}[E] > 0$$

$$\Rightarrow \frac{\mathbb{P}[E\cap F]}{\mathbb{P}[E]\cdot\mathbb{P}[F]} > 1 \qquad / \cdot \mathbb{P}[F]$$

$$\Rightarrow \boxed{\frac{\mathbb{P}[E\cap F]}{\mathbb{P}[E]}} > \mathbb{P}[F]$$

By def'n.

$$\mathbb{P}[F|E] > \mathbb{P}[F]$$

T

□

**Problem 1.2.** Let $A$ and $B$ be events such that $\mathbb{P}[A] = 1/2, \mathbb{P}[B] = 1/3$ and $\mathbb{P}[A \cap B] = 1/4$. Calculate the following probabilities:

(i) $\mathbb{P}[A \cup B]$
(ii) $\mathbb{P}[A|B]$
(iii) $\mathbb{P}[B|A]$
(iv) $\mathbb{P}[A^c|B^c]$

Inclusion-Exclusion Formula

(i) $\mathbb{P}[A\cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A\cap B]$

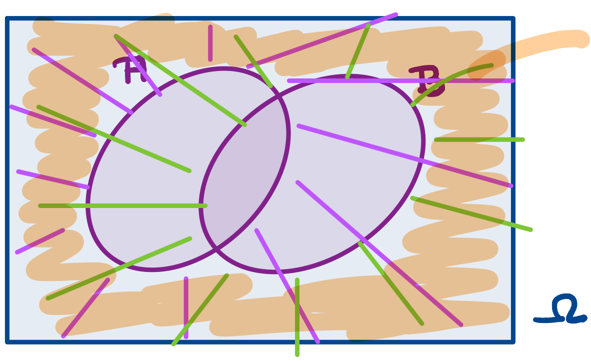
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{6+4-3}{12} = \frac{7}{12}$$

(ii) $$\mathbb{P}[A|B] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

def'n

(iii) $$\mathbb{P}[B|A] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[A]} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(iv) $$\mathbb{P}[A^c|B^c] = \frac{\mathbb{P}[A^c\cap B^c]}{\mathbb{P}[B^c]} = \frac{1-\mathbb{P}[A\cup B]}{1-\mathbb{P}[B]} = \frac{\frac{5}{12}}{\frac{2}{3}} = \frac{5}{8}$$ $\square$




$(A\cup B)^c = A^c \cap B^c$

similarly

$(A\cap B)^c = A^c \cup B^c$

## Multiplication Formula for Conditional Probability.

$$\boxed{\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2 \mid A_1]}$$

We can extend this formula using mathematical induction:

$$\boxed{\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2 \mid A_1] \cdot \mathbb{P}[A_3 \mid A_1 \cap A_2] \cdot \ldots \cdot \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}]}$$

**Example.** A lot contains 12 items out of which 4 are defective. Three items are drawn from the lot, one after the other. Find the probability that all three are non-defective.

$\longrightarrow$: Method I.

$A_i = \{\text{item } \# i \text{ is non-defective}\}$ w/ $i = 1, 2, 3$

$$\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1] \cdot \mathbb{P}[A_2 \mid A_1] \cdot \mathbb{P}[A_3 \mid A_1 \cap A_2]$$

$$= \frac{\cancel{8}\,2}{\cancel{12}\,\cancel{3}} \cdot \frac{7}{11} \cdot \frac{\cancel{6}\,\cancel{3}}{\cancel{10}\,5} = \frac{14}{55}$$

Method II.

$$\text{prob.} = \frac{\#\text{ of non-defective trios}}{\#\text{ of trios total}} = \frac{\binom{8}{3}}{\binom{12}{3}} = \frac{\frac{8 \cdot 7 \cdot 6}{\cancel{3!}}}{\frac{12 \cdot 11 \cdot 10}{\cancel{3!}}}$$

$\square$

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$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k!}$$