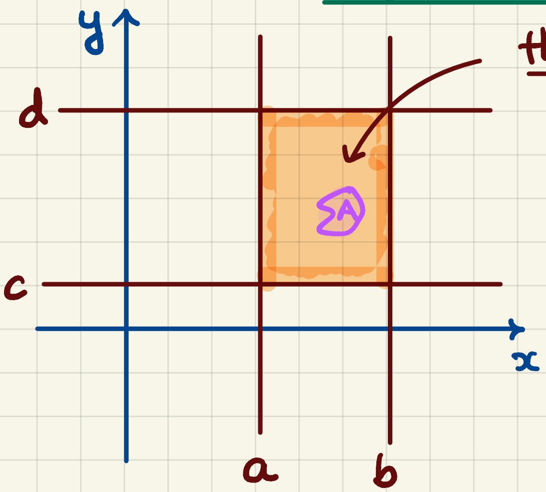
M362K: April 29th, 2024.

Chapter 5.

Section 5.1. Uniform Densities.



the rectangular region

$$\boxed{B = [a,b] \times [c,d]}$$

"Throwing fair (unbiased) darts @ the region B."

Position of the dart is determined by a pair of coordinates:

$$\boxed{(X,Y) \text{ a RANDOM PAIR}}$$

We introduce the joint density function of (X,Y) as:

$$f_{X,Y}(x,y) = \begin{cases} \text{const} & \text{for } (x,y) \in B \\ 0 & \text{for } (x,y) \notin B \end{cases}$$

for

$$\text{const} = \frac{1}{\text{area}(B)} = \frac{1}{(b-a)(d-c)}$$

Let $A \subseteq B$:

$$\mathbb{P}[(X,Y) \in A] = \frac{\text{area}(A)}{\text{area}(B)} = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

Example. A bird will land @ random in a square area w/ each side of length 2.
What is the probability that the bird lands in a trap which is circular w/ radius 3/4?

$\longrightarrow$!

$$\frac{\text{area (trap)}}{\text{area (field)}} = \frac{\left(\frac{3}{4}\right)^2 \pi}{2^2} = \frac{9}{64}\pi$$

□

Def'n. We say that (X,Y) are jointly continuous w/ pdf

$$f_{X,Y}: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

if

$$\boxed{\mathbb{P}[(X,Y)\in A] = \iint_A \underline{f_{X,Y}(x,y)}\,dx\,dy}$$

Def'n. Random variables X and Y are independent, if

$$\boxed{\mathbb{P}[X\in A_1, Y\in A_2] = \mathbb{P}[X\in A_1]\cdot\mathbb{P}[Y\in A_2]}$$

for all $A_1, A_2 \subseteq \mathbb{R}$

Recall: For X and Y jointly discrete w/ joint pmf $p_{X,Y}$, they are independent iff

$$\boxed{p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)}$$

Def'n. Marginal Densities.

$$\underline{f_X(x)} = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \qquad \text{for all } x\in\mathbb{R}$$

$$\underline{f_Y(y)} = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \qquad \text{for all } y\in\mathbb{R}$$

Then, jointly continuous X and Y are independent iff

$$\boxed{f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)} \qquad \text{for all } (x,y)\in\mathbb{R}^2$$

Note: X and Y independent

$$\mathbb{E}[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,\boxed{f_{X,Y}(x,y)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy = \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

Independent Normals.

Imagine repeated measurements:

$$X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$$

the actual value we're measuring

We define the sample mean:

$$\boxed{\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \sim N\left(\mu, \frac{\sigma^2}{n}\right)}$$