M362K: April 26th, 2024.

Linear Change of Variable for Densities.

Say: X is a continuous r.v. w/ pdf f_X

Define: $\boxed{Y = aX + b}$ for some constants $a \neq 0$ and b

For every $y \in \mathbb{R}$:

$$F_Y(y) = \mathbb{P}[Y \leq y] = $$
$$= \mathbb{P}[aX + b \leq y]$$
$$= \mathbb{P}[aX \leq y - b]$$

Case 1. $a > 0$

$$F_Y(y) = \mathbb{P}\left[X \leq \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y)$$
$$= \frac{1}{a} F_X'\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

chain rule

Case 2. $a < 0$

$$F_Y(y) = \mathbb{P}\left[X \geq \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow \quad f_Y(y) = F_Y'(y) = -\frac{1}{a} F_X'\left(\frac{y-b}{a}\right) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Example. The monthly profit of Company I can be modeled by a continuous random variable w/ pdf f.
Company II has a monthly profit that is twice that of Company I.

What is the pdf of the monthly profit of Company II?

$\longrightarrow$: X... profit of Company I
Y... profit of Company II
$\Rightarrow \quad Y = 2X$

$\Rightarrow$ With the above formula : $a=2$ and $b=0$

$$\Rightarrow \boxed{f_Y(y) = \frac{1}{2} f\left(\frac{y}{2}\right)}$$

□

Problem. Let T denote the time in minutes for a customer service representative to respond to 10 phone calls.
Assume that T is uniformly distributed on the interval from 8 minutes to 12 minutes.

Let R denote the average rate, in customers per minute, @ which the representative responds to inquiries.

Find the pdf of the r.v. R.

$\rightarrow$: T... # of minutes (spent on) PER 10 customers
R... # of customers PER minute

} inversely proportional

$$\boxed{R = \frac{10}{T}} \quad w/ \quad \boxed{T \sim U(8,12)}$$

$f_T(t) = \frac{1}{4}$ for $t \in (8,12)$

$$\text{Support}(R) = \left(\frac{10}{12}, \frac{10}{8}\right)$$

For $r \in \left(\frac{10}{12}, \frac{10}{8}\right)$

$$F_R(r) = \mathbb{P}[R \le r] = \mathbb{P}\left[\frac{10}{T} \le r\right] =$$

$$= \mathbb{P}\left[\frac{10}{r} \le T\right] = 1 - \mathbb{P}\left[T \le \frac{10}{r}\right]$$

$$= 1 - F_T\left(\frac{10}{r}\right)$$

$$\Rightarrow f_R(r) = +(+1)\frac{10}{r^2}\boxed{f_T\left(\frac{10}{r}\right)}$$

$$f_R(r) = \frac{10}{r^2} \cdot \frac{1}{4} = \frac{5}{2r^2}$$

□

One-to-One Change of Variables.

Say, X is a continuous r.v. w/ pdf f_X on the range (a,b)
(a can be $-\infty$, b can be too)

Define, $Y = g(X)$ w/ $g : (a,b) \longrightarrow \mathbb{R}$
such that g is strictly monotone
(i.e., strictly increasing or strictly decreasing).

Then, the support of Y is

either $[g(a), g(b)]$ for $g \uparrow$
or $[g(b), g(a)]$ for $g \downarrow$

We also have the following formula for the pdf of Y

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x)$$

w/ $y = g(x)$

where we have substitute $x = g^{-1}(y)$ on the right-hand side to get the expression for $f_Y(y)$ just in terms of y.