M362K: April 22nd, 2024.

Section 4.2.

The Exponential Distribution.

A r.v. T is said to have the exponential distribution with rate λ, where λ is a positive parameter, if its pdf is of this form

$$\boxed{f_T(t) = \lambda e^{-\lambda \cdot t} \quad t \geq 0}$$

The cumulative dist'n f'tion of T is, then,

$$F_T(t) = \int_0^t f_T(u)\,du = \int_0^t \lambda e^{-\lambda \cdot u}\,du =$$

$$= \lambda \cdot \int_0^t e^{-\lambda u}\,du = \cancel{\lambda} \cdot \left(-\frac{1}{\cancel{\lambda}}\right) e^{-\lambda u} \Big|_{u=0}^{t}$$

$$= 1 - e^{-\lambda \cdot t} \qquad \text{for } t \geq 0$$

Def'n. For a r.v. X w/ cdf F_X, its survival function is defined as

$$\boxed{S_X(x) = \mathbb{P}[X > x] = 1 - F_X(x) \quad \text{for all } x \in \mathbb{R}}$$

Properties.

- W/out proof: $\boxed{\mathbb{E}[T] = SD[T] = \frac{1}{\lambda}}$
- $\boxed{S_T(t) = e^{-\lambda t}}$

Problem. The amt of time, in hrs, that a computer functions before breaking down is a continuous r.v. w/ pdf

$$f_T(t) = \begin{cases} \boxed{\varkappa e^{-\frac{t}{100}}} & \text{for } t \geq 0 \\ 0 & x < 0 \end{cases}$$

(i) What is the probability that the computer will function for less than 100 hrs?

$\longrightarrow$: $T \sim \text{Exponential}\left(\lambda = \frac{1}{100}\right) \Rightarrow \varkappa = \frac{1}{100}$

$$\mathbb{P}[T \leq 100] = F_T(100) = 1 - e^{-\frac{100}{100}} = 1 - e^{-1} = 0.632$$

(ii) What is the probability that the computer will function between 50 and 150 hrs before breaking down?

$\longrightarrow$: $\mathbb{P}[50 < T < 150] = F_T(150) - F_T(50)$

$$= \cancel{1} - e^{-\frac{150}{100}} - \left(\cancel{1} - e^{-\frac{50}{100}}\right) =$$

$$= e^{-\frac{50}{100}} - e^{-\frac{150}{100}} = S_T(50) - S_T(150)$$

$$= e^{-0.5} - e^{-1.5}$$

$\square$

Half·Life.

Let m denote the half·life of $T \sim Exp(\lambda)$, i.e., let m satify:

$$\underbrace{\mathbb{P}[T \leq m]} = \frac{1}{2} = \underbrace{\mathbb{P}[T \geq m]} \quad \text{(T continuous)}$$

$$\frac{1}{2} = S_T(m)$$

$$\frac{1}{2} = e^{-\lambda \cdot m}$$

$$+\ln(2) = \ln\left(\frac{1}{2}\right) = +\lambda \cdot m$$

$$\boxed{m = \frac{\ln(2)}{\lambda}}$$

In other words $\frac{\ln(2)}{\lambda}$ is the median of $Exp(\lambda)$.

Note: In the previous problem,

- the half·life would be $m = \dfrac{\ln(2)}{\frac{1}{100}} = 100\ln(2)$
- the mean would be $\dfrac{1}{\lambda} = 100$

Memoryless Property.

Let $T \sim Exp(\lambda)$. Let $s,t \geq 0$:

$$\mathbb{P}[T > s+t \mid T > t] = \frac{\mathbb{P}[T>s+t, T>t]}{\mathbb{P}[T>t]} = \frac{\mathbb{P}[T>s+t]}{\mathbb{P}[T>t]} =$$

by def'n of conditional probability

$$= \frac{e^{-\lambda(s+\cancel{t})}}{\cancel{e^{-\lambda t}}} = e^{-\lambda s} = S_T(s) = \mathbb{P}[T>s]$$

survival f'tion

28. The number of days that elapse between the beginning of a calendar year and the moment a high-risk driver is involved in an accident is exponentially distributed. An insurance company expects that 30% of high-risk drivers will be involved in an accident during the first 50 days of a calendar year.

Calculate the portion of high-risk drivers are expected to be involved in an accident during the first 80 days of a calendar year.

(A) 0.15
(B) 0.34
(C) 0.43
(D) 0.57
(E) 0.66

T… # of days before a h·r driver has an accident

$\mathbb{P}[T \le 50] = 0.3 \Rightarrow \mathbb{P}[T > 50] = 0.7 \Rightarrow \underbrace{e^{-\lambda \cdot 50} = 0.7}_{e^{-10\lambda} = (0.7)^{1/5}}$

$\mathbb{P}[T \le 80] = 1 - e^{-80\lambda} = 1 - (e^{-10\lambda})^8 = \boxed{1 - (0.7)^{8/5}}$

29. An actuary has discovered that policyholders are three times as likely to file two claims as to file four claims.

The number of claims filed has a Poisson distribution.

Calculate the variance of the number of claims filed.

(A) $\dfrac{1}{\sqrt{3}}$
(B) 1
(C) $\sqrt{2}$
(D) 2
(E) 4

30. A company establishes a fund of 120 from which it wants to pay an amount, *C*, to any of its 20 employees who achieve a high performance level during the coming year. Each employee has a 2% chance of achieving a high performance level during the coming year. The events of different employees achieving a high performance level during the coming year are mutually independent.

Calculate the maximum value of *C* for which the probability is less than 1% that the fund will be inadequate to cover all payments for high performance.

(A) 24
(B) 30
(C) 40
(D) 60
(E) 120

266. Claim amounts are independent random variables with probability density function

$$f(x) = \begin{cases} \frac{10}{x^2}, & \text{for } x > 10 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the probability that the largest of three randomly selected claims is less than 25.

(A) $\frac{8}{125}$
(B) $\frac{12}{125}$
(C) $\frac{27}{125}$
(D) $\frac{2}{5}$
(E) $\frac{3}{5}$

267. The lifetime of a certain electronic device has an exponential distribution with mean 0.50.

Calculate the probability that the lifetime of the device is greater than 0.70, given that it is greater than 0.40.

(A) 0.203
(B) 0.247
(C) 0.449
(D) 0.549
(E) 0.861

$\longrightarrow$: Lifetime $T \sim Exp(\lambda = 2)$

$$\mathbb{P}[T > 0.7 \mid T > 0.4] \underset{\text{memoryless property}}{=} \mathbb{P}[T > 0.3] = S_T(0.3)$$

$$= e^{-2(0.3)}$$

$$= e^{-0.6}$$

Example. $\boxed{4.5.3.}$ Minimum of Independent Exponential r.v.s.

Let $X_1, X_2, \dots, X_n$ be independent r.v.s

w/ $X_i \sim Exp(\lambda_i)$ for $i = 1, \dots, n$.

Define: $X_{min} = \min(X_1, X_2, \dots, X_n)$.

Let's find the cdf of X_{min}.

For every $x > 0$: $F_{X_{min}}(x) = \mathbb{P}[X_{min} \le x] = 1 - \mathbb{P}[X_{min} > x]$

$$= 1 - \mathbb{P}[X_1 > x, X_2 > x, \dots, X_n > x]$$

independence

$$= 1 - S_{X_1}(x) \cdot S_{X_2}(x) \cdots S_{X_n}(x)$$

$$= 1 - e^{-\lambda_1 x} \cdot e^{-\lambda_2 \cdot x} \cdots e^{-\lambda_n \cdot x}$$

$$= 1 - e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \cdot x}$$

$$\boxed{X_{min} \sim Exp(\lambda = \lambda_1 + \cdots + \lambda_n)}$$

248. Let X be a random variable that takes on the values –1, 0, and 1 with equal probabilities.

Let $Y = X^2$.

Which of the following is true?

(A) $\text{Cov}(X, Y) > 0$; the random variables X and Y are dependent.
(B) $\text{Cov}(X, Y) > 0$; the random variables X and Y are independent.
(C) $\text{Cov}(X, Y) = 0$; the random variables X and Y are dependent.
(D) $\text{Cov}(X, Y) = 0$; the random variables X and Y are independent.
(E) $\text{Cov}(X, Y) < 0$; the random variables X and Y are dependent.

249. Losses follow an exponential distribution with mean 1. Two independent losses are observed.

Calculate the expected value of the smaller loss.

(A) 0.25
(B) 0.50
(C) 0.75
(D) 1.00
(E) 1.50

250. A delivery service owns two cars that consume 15 and 30 miles per gallon. Fuel costs 3 per gallon. On any given business day, each car travels a number of miles that is independent of the other and is normally distributed with mean 25 miles and standard deviation 3 miles.

Calculate the probability that on any given business day, the total fuel cost to the delivery service will be less than 7.

(A) 0.13
(B) 0.23
(C) 0.29
(D) 0.38
(E) 0.47