M362K: April 19th, 2024.

Problem from Sample Exam.

Consider two independent r.v.s X and Y w/ cdfs F_X and F_Y, resp.

Define:

$$U := \boxed{\max(X,Y)}$$

and

$$V := \boxed{\min(X,Y)}$$

Express F_U and F_V in terms of F_X and F_Y.

$\longrightarrow$: By def'n: for all $x \in \mathbb{R}$

$$\begin{aligned}
F_U(x) &= \mathbb{P}[U \leq x] \\
&= \mathbb{P}[\max(X,Y) \leq x] = \\
&= \mathbb{P}[X \leq x, Y \leq x] = \quad \text{independence} \\
&= \underbrace{\mathbb{P}[X \leq x]}\cdot\underbrace{\mathbb{P}[Y \leq x]} \\
&= F_X(x) \cdot F_Y(x)
\end{aligned}$$

By def'n: for all $x \in \mathbb{R}$

$$\begin{aligned}
F_V(x) &= \mathbb{P}[V \leq x] \\
&= \mathbb{P}[\min(X,Y) \leq x] \qquad \mathbb{P}[E] = 1 - \mathbb{P}[E^c] \\
&= 1 - \mathbb{P}[\min(X,Y) > x] = \\
&= 1 - \mathbb{P}[X > x, Y > x] = \quad \text{independence} \\
&= 1 - \underbrace{\mathbb{P}[X > x]}\cdot\underbrace{\mathbb{P}[Y > x]} \\
&= 1 - (1 - F_X(x))(1 - F_Y(x)) \\
&= \cancel{1} - \cancel{1} + F_X(x) + F_Y(x) - F_X(x)F_Y(x) \\
&= F_X(x) + F_Y(x) - F_X(x) \cdot F_Y(x)
\end{aligned}$$

$\square$

More about the Normal Dist'n.

Recall: $Z \sim N(0,1)$

Any normal $X \sim N(\mu, \sigma^2)$ can be expressed as

$$\boxed{X = \mu + \sigma \cdot Z} \quad \text{for some } Z \sim N(0,1)$$

- $\mathbb{E}[X] = \mathbb{E}[\mu + \sigma \cdot Z] \underset{\substack{\uparrow \\ \text{linearity}}}{=} \mu + \sigma \cdot \underbrace{\mathbb{E}[Z]}_{=0} = \mu$
- $\mathrm{Var}[X] = \mathrm{Var}[\underset{\text{shift}}{\mu} + \sigma \cdot Z] = \sigma^2 \cdot \underbrace{\mathrm{Var}[Z]}_{=1} = \sigma^2$

Problem. Let X be normally distributed w/ mean μ and std dev σ. What is the probability that X falls within $1/10$ of the std dev from its mean?

$\longrightarrow$: $X \sim \mathrm{Normal}(\text{mean} = \mu, \text{sd} = \sigma)$

$$\mathbb{P}\left[\,|X - \mu| < \tfrac{1}{10}\sigma\right] = ?$$

$$\mathbb{P}\left[-\tfrac{1}{10}\sigma < X - \mu < \tfrac{1}{10}\sigma\right] =$$

$$= \mathbb{P}\left[-0.1 < \underset{\sim N(0,1) \sim Z}{\frac{X-\mu}{\sigma}} < 0.1\right] =$$

$$= \mathbb{P}[-0.1 < Z < 0.1] =$$

$$= \Phi(0.1) - \underbrace{\Phi(-0.1)}_{1 - \Phi(0.1)} = 2\Phi(0.1) - 1 = 2 \cdot 0.5398 - 1$$

$$= 1.0796 - 1$$

$$= 0.0796 \quad \square$$

The Central Limit Theorem.

Let $\{X_1, X_2, \ldots\}$ be a sequence of i.i.d. r.v.s w/ $\mu_X = \mathbb{E}[X_1] < \infty$ and $\sigma_X^2 = \mathrm{Var}[X_1] < \infty$.

Define for all $n \in \mathbb{N}$:

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$\boxed{\frac{S_n - n\cdot\mu_X}{\sigma_X\sqrt{n}} \xRightarrow[n\to\infty]{\mathcal{D}} N(0,1)}$$

Problem. Automobile losses are independent and uniformly dist'd between 0 and 15,000.
Calculate the (approximate) probability that the total amount for 200 losses lies between 1,450,000 and 1,700,000.

$\longrightarrow$: $X_i \sim U(0, 15000)$ $\quad i = 1, \ldots, 200$

$$S_{200} = X_1 + X_2 + \cdots + X_{200}$$

We need $\boxed{\mathbb{P}[1.45\cdot 10^6 < S_{200} < 1.7\cdot 10^6] = ?}$

$\mu_X = \mathbb{E}[X_1] = 7500$

$\mathrm{Var}[X_1] = \frac{(15000)^2}{12} = 18\,750\,000$

$\sigma_X = 4330.127$

In general: $X \sim U(a,b)$

$\mathbb{E}[X] = \frac{a+b}{}$

$\mathrm{Var}[X] = \frac{(b-a)^2}{12}$

$$\mathbb{P}\left[\frac{1.45\cdot 10^6 - 7500\cdot 200}{4330.127\cdot\sqrt{200}} < \overset{\sim N(0,1)}{Z} < \frac{1.7\cdot 10^6 - 7500\cdot 200}{4330.127\sqrt{200}}\right] =$$

$$\cong \mathbb{P}[-0.82 < Z < 3.27] = \Phi(3.27) - \Phi(-0.82)$$

$$= 0.9995 - 0.2061 = 0.7934$$

$\square$

Section 4.2.

Exponential Distribution.

A random variable T has the exponential distribution with rate λ, written as

$$\boxed{T \sim Exp(\lambda)}$$

If T has the pdf given by $\boxed{f_T(t) = \lambda e^{-\lambda t} \quad \text{for } t \geq 0}$

Q: What is the cdf of T?

$\longrightarrow$: $F_T(t) = \int_0^t f_T(u)\,du = \int_0^t \lambda e^{-\lambda \cdot u}\,du = \cancel{\lambda} \cdot \left(-\frac{1}{\cancel{\lambda}}\right) e^{-\lambda u}\Big|_{u=0}^{t}$

$F_T(t) = 1 - e^{-\lambda t}$

Def'n. The survival function of any random variable X is $S_X : \mathbb{R} \to [0,1]$ given by $\boxed{S_X(x) = 1 - F_X(x)}$ for all $x \in \mathbb{R}$

Note: For $T \sim Exp(\lambda)$: $\boxed{S_T(t) = \mathbb{P}[T > t] = e^{-\lambda t}}$

Without proof: $\boxed{\mathbb{E}[T] = SD[T] = \frac{1}{\lambda}}$