M362K: April 17th, 2024.

More on Continuous Random Variables.

Def'n. Let X be a continuous random variable w/ a probability density function f_X:
The expectation of X is defined as:

$$\boxed{\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx}$$

If the integral is absolutely convergent

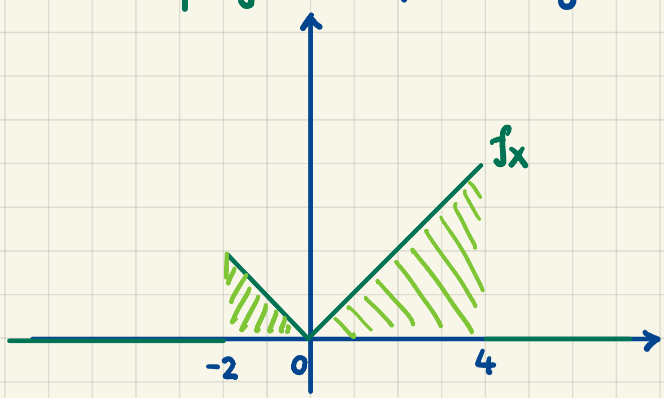
Problem. Let X be a continuous r.v. w/ pdf proportional to

$$|x| \quad \text{for} \quad \boxed{-2 \le x \le 4,}$$

and otherwise equal to zero.
Calculate the expected value of X.

$\longrightarrow$: Support $(X) = [-2, 4]$

The pdf is of the form $f_X(x) = \overset{?}{\kappa} |x|$ for $-2 \le x \le 4$



area of the shaded region:

$2 + 8 = 10$

$\Rightarrow \kappa = \frac{1}{10}$

$$\mathbb{E}[X] = \int_{-2}^{4} x f_X(x)\,dx = \int_{-2}^{0} x \cdot \left(-\frac{x}{10}\right) dx + \int_{0}^{4} x \cdot \frac{x}{10}\,dx$$

$$= \frac{1}{10}\left(-\int_{-2}^{0} x^2\,dx + \int_{0}^{4} x^2\,dx\right)$$

$$= \frac{1}{10}\left(-\frac{1}{3}x^3\Big|_{x=-2}^{0} + \frac{1}{3}x^3\Big|_{x=0}^{4}\right)$$

$$= \frac{1}{10}\left(-\frac{8}{3} + \frac{64}{3}\right) = \frac{56}{30} = \frac{28}{15} \quad \square$$

$$-\frac{1}{3}\left(0^3 - (-2)^3\right)$$

Defn.

$$\boxed{\mathbb{E}[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx}$$ If the integral exists

In particular, the second moment is

$$\boxed{\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx}$$

Recall: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\Rightarrow \text{SD}[X] = \sqrt{\text{Var}[X]}$

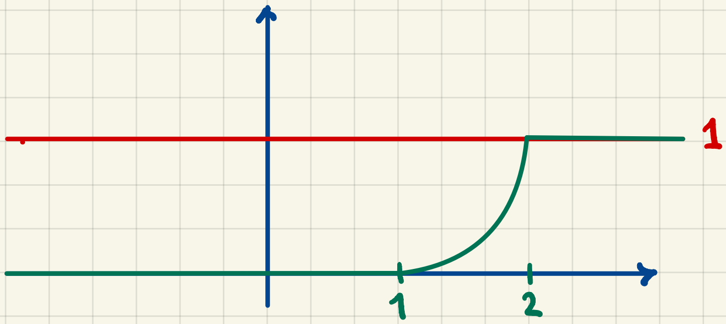
All the properties of the expectation and variance will be INHERITED from discrete distributions!

Problem. Consider a r.v. X whose cdf is given by

$$F_X(x) = \begin{cases} 0 & \text{for } x < 1 \\ (x-1)^2 & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Q: Is X a continuous r.v.?

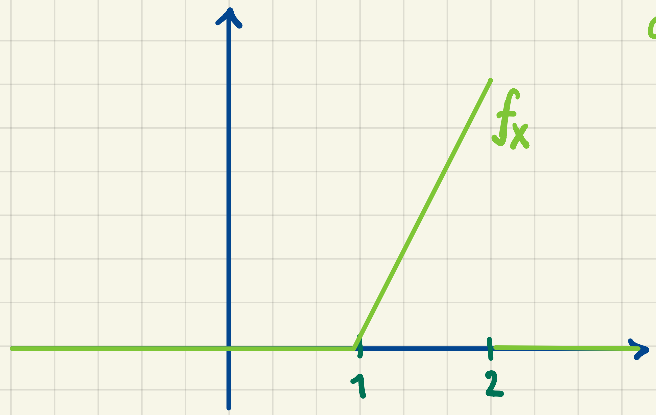
$\longrightarrow$:



$\Rightarrow$ X is continuous

The pdf of X is

$f_X(x) = F_X'(x) = \boxed{2(x-1)}$ for $1 \leq x \leq 2$ and 0 otherwise



Q: $\mathbb{E}[X] = ?$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$= \int_1^2 x \cdot 2\cdot(x-1)\,dx = 2\int_1^2 (x^2 - x)\,dx$$

$$= 2\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{x=1}^{2}\right.$$

$$= 2\left(\frac{8}{3} - \frac{4}{2} - \left(\frac{1}{3} - \frac{1}{2}\right)\right) = 2\left(\frac{7}{3} - \frac{3}{2}\right) =$$

$$= 2 \cdot \frac{14-9}{6} = \boxed{\frac{5}{3}}$$

Q: $Var[X] = ?$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

$$= \int_1^2 x^2 \cdot 2(x-1)\,dx = 2\int_1^2 (x^3 - x^2)\,dx$$

$$= 2\left(\frac{x^4}{4} - \frac{x^3}{3}\Big|_{x=1}^{2}\right.$$

$$= 2\left(\frac{16}{4} - \frac{8}{3} - \left(\frac{1}{4} - \frac{1}{3}\right)\right)$$

$$= 2\left(\frac{15}{4} - \frac{7}{3}\right) = 2 \cdot \frac{45-28}{12} = \frac{17}{6}$$

$$Var[X] = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \frac{1}{18}$$

$\square$

Uniform Distribution.

Unit (standard) uniform dist'n is denoted by $\boxed{U \sim U(0,1)}$ and determined by density: $\boxed{f_U(x) = 1 \quad \text{for } 0 < x < 1}$

In general, a uniform dist'n on any interval: $\boxed{U(a,b)}$

Let $X \sim U(a,b)$.

Then,

$$X-a \sim U(0, b-a)$$

$$\boxed{\frac{X-a}{b-a} \sim U(0,1)}, \text{ i.e., } \boxed{X = a + (b-a)\cdot U} \checkmark$$

for some $U \sim U(0,1)$

Q: $\mathbb{E}[U] = ?$

$$\mathbb{E}[U] = \int_0^1 x \cdot 1\, dx = \frac{x^2}{2}\Big|_{x=0}^{1} = \frac{1}{2} \checkmark$$

$$\Rightarrow \mathbb{E}[X] = \mathbb{E}[a + (b-a)\cdot U] = a + (b-a)\cdot\mathbb{E}[U]$$

$$= a + (b-a)\cdot\frac{1}{2} = \frac{a+b}{2}$$

Q: $\mathrm{Var}[U] = ?$

$$\mathbb{E}[U^2] = \int_0^1 x^2 \cdot 1\, dx = \frac{x^3}{3}\Big|_{x=0}^{1} = \frac{1}{3}$$

$$\mathrm{Var}[U] = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

$$\Rightarrow \mathrm{Var}[X] = \mathrm{Var}[a + (b-a)\cdot U] = \mathrm{Var}[(b-a)\cdot U] =$$

$$= (b-a)^2 \cdot \mathrm{Var}[U] = \frac{(b-a)^2}{12}$$

The Normal Distribution.

The standard normal r.v. $Z \sim N(0,1)$ has the pdf

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

Any normal random variable $X \sim N(\mu, \sigma^2)$ can be written as

$$\boxed{X = \mu + \sigma \cdot Z} \quad \text{for } Z \sim N(0,1)$$

Note: $\mathbb{E}[Z] = \int_{-\infty}^{+\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0$ $\qquad \boxed{\mathrm{Var}[Z] = 1}$

odd function