M362K: April 15th, 2024.

Chapter IV

Continuous Distributions.

Example. Uniform Distribution.

Any value between c and d is possible ($c<d$).

Assume: probability of hitting an interval is proportional to its length.

$\Rightarrow$ There is a constant $\varkappa > 0$ such that for any α and β such that $c \leq \alpha \leq \beta \leq d$, we have

$$\boxed{\mathbb{P}[(\alpha, \beta)] = \varkappa \cdot (\beta - \alpha)}$$

In particular, for $\alpha = c$ and $\beta = d$, we get

$$\mathbb{P}[(c,d)] = \varkappa \cdot (d-c) = 1 \quad \Rightarrow \quad \boxed{\varkappa = \frac{1}{d-c}}$$

Our uncountable outcome space: $\Omega = (c,d)$

w/ probability dist'n: $\boxed{\mathbb{P}[(\alpha,\beta)] = \frac{\beta-\alpha}{d-c}}$ for $c \leq \alpha \leq \beta \leq d$

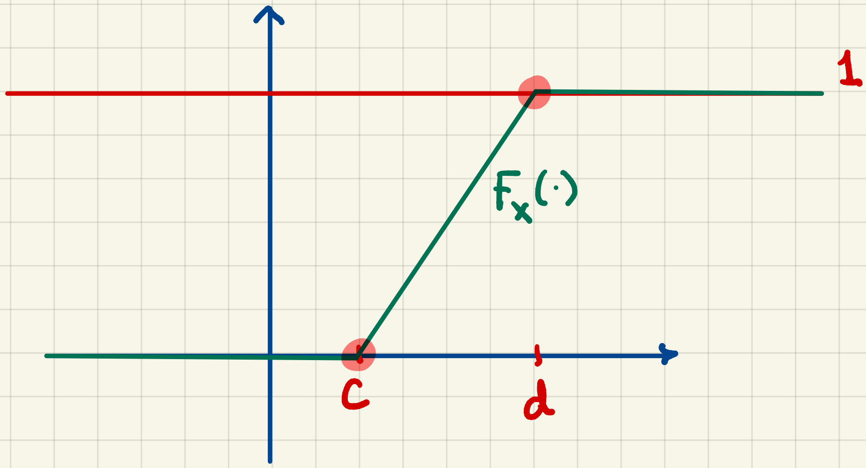
This is sufficient to figure out the probability of any other "nice" subset of (c,d), i.e., any event.

We introduce the uniform random variable X on Ω so that $\boxed{X(\omega) = \omega}$

Q: What is the cdf of X?

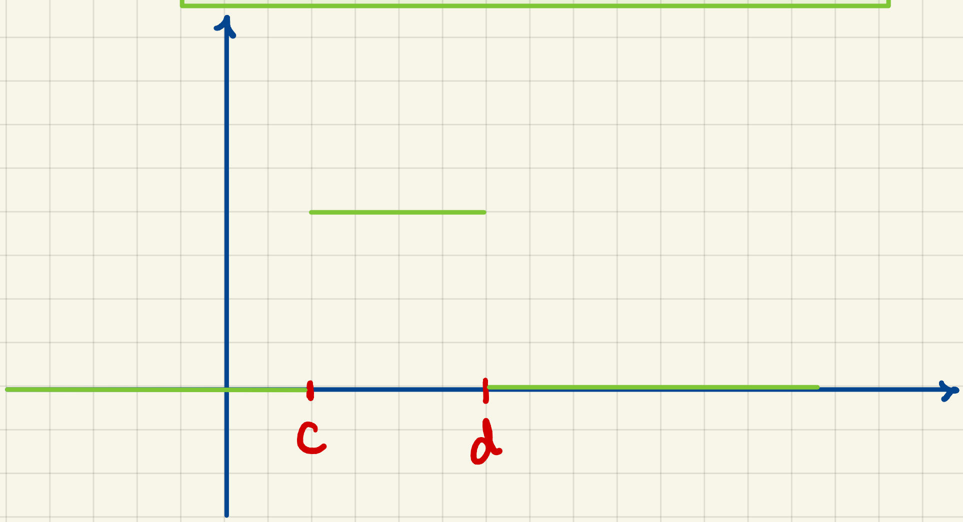
$\longrightarrow$: $F_X(x) = \mathbb{P}[X \leq x]$ for $x \in \mathbb{R}$

$$\boxed{F_X(x) = \begin{cases} 0 & \text{if } x \leq c \\ \frac{x-c}{d-c} & \text{if } c < x < d \\ 1 & \text{if } x \geq d \end{cases}}$$



Note: F_X is continuous
and differentiable almost everywhere
$\Rightarrow$ We can find the derivative F_X' where it exists

$$\boxed{F_X'(x) = \begin{cases} 0 & x<c \\ \frac{1}{d-c} & c<x<d \\ 0 & x>d \end{cases}}$$



This is the uniform probability density function.

Def'n. Any function $f:\mathbb{R}\to\mathbb{R}$ such that

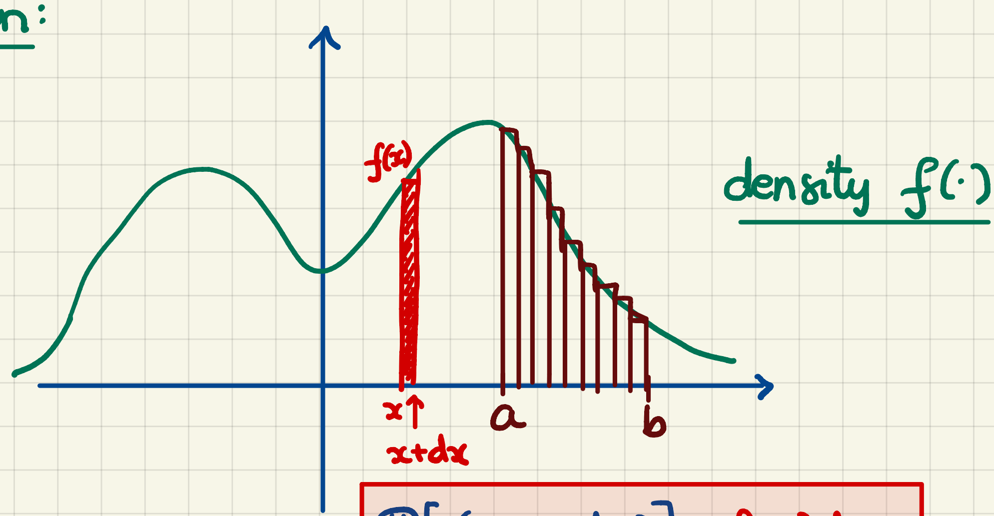
- $f(x)\geq 0$ for all $x\in\mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) = 1$

is called a (probability) density function.

e.g., $f = \varphi$ w/ φ the standard normal density f'tion, i.e.,

$$\boxed{\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } z \in \mathbb{R}}$$

Interpretation:



$$\boxed{\mathbb{P}[(x, x+dx)] = f(x)dx}$$

$$\boxed{\mathbb{P}[(a,b)] = \int_a^b f(x)\,dx}$$

Def'n. A continuous random variable is a "nice" function

$$X : \Omega \longrightarrow \mathbb{R}$$

such that

$$\mathbb{P}[a < X \leq b] = \int_a^b f_X(x)\,dx$$

for all $a < b$ and for some density function f_X.

We say that f_X is the (probability) density f'tion of X.

Properties:
- $\mathbb{P}[X = a] = 0$
- The cumulative dist'n ftion:

$$F_X(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} f_X(u)\,du$$

which means that $\boxed{F_X'(x) = f_X(x)}$ wherever the derivative exists

Problem. The lifetime of a machine part has a continuous dist'n on the interval $(0,40)$ w/ the pdf proportional to

$$\frac{1}{(10+x)^2}.$$

Calculate the probability that the lifetime of the machine part is less than 6.

$\longrightarrow$: X... the lifetime

$$f_X(x) = \varkappa(10+x)^{-2} \quad \text{for } x \in (0,40)$$

We want $\mathbb{P}[X<6] = \mathbb{P}[0<X<6] =$

$$= \int_0^6 \frac{\varkappa}{(10+x)^2}\,dx$$

We get $\varkappa$ using:

$$\int_0^{40} f_X(x)\,dx = 1$$

$$1 = \varkappa \int_0^{40} (10+x)^{-2}\,dx = \varkappa \cdot (-1) \cdot (10+x)^{-1} \Big|_{x=0}^{40}$$

$$= \varkappa \left(\frac{1}{10} - \frac{1}{50}\right)$$

$$= \varkappa \cdot \frac{4}{50} = \varkappa \cdot \frac{2}{25} \Rightarrow \boxed{\varkappa = 12.5}$$

$$\mathbb{P}[X<6] = 12.5(-1)(10+x)^{-1} \Big|_{x=0}^{6}$$

$$= 12.5\left(\frac{1}{10} - \frac{1}{16}\right) =$$

$$= 12.5\,\frac{8-5}{80} = \frac{12.5(3)}{80} = \frac{375}{800} = \frac{75}{160} = \frac{15}{32}$$

$\square$

Problem. The loss due to fire is modeled by a r.v. X w/ pdf of the form

$$f_X(x) = \begin{cases} \varkappa(20-x) & \text{for } 0<x<20 \\ 0 & \text{otherwise} \end{cases}$$

for some constant $\varkappa > 0$.

Given that a fire loss exceeds 8, what is the probability that it exceeds 16?

$\longrightarrow$: X... fire loss

$$\mathbb{P}[X>16 \mid X>8] = \cancel{?} \frac{\mathbb{P}[X>16, X>8]}{\mathbb{P}[X>8]} = \frac{\mathbb{P}[X>16]}{\mathbb{P}[X>8]}$$

$\mathbb{P}[X>d]$ for any $d \in (0,20)$

$$\mathbb{P}[X>d] = \int_d^{20} \varkappa(20-x)\,dx = \varkappa \int_d^{20} (20-x)\,dx$$

$$\mathbb{P}[X>d] = \varkappa\left(20(20-d) - \frac{x^2}{2}\Big|_d^{20}\right)$$

$$= \varkappa\left(400 - 20d - \frac{400}{2} + \frac{d^2}{2}\right)$$

$$= \varkappa\left(200 - 20d + \frac{d^2}{2}\right)$$

$$\mathbb{P}[X>16 \mid X>8] = \frac{\cancel{\varkappa}\left(200 - 20(16) + \frac{16^2}{2}\right)}{\cancel{\varkappa}\left(200 - 20(8) + \frac{8^2}{2}\right)} = \frac{8}{72} = \frac{1}{9}$$

$\square$