M362K: April 12th, 2024.

Section 3.5.

The Poisson Distribution.

Recall. An approximation to Binomial(n, p) w/ n "large" and p either close to 0 or close to 1

$\boxed{\text{Rare events.}}$

For all $k \in \mathbb{N}_0$:

$$\mathbb{P}[k \text{ successes in } n \text{ trials}] \approx \boxed{e^{-\mu} \cdot \frac{\mu^k}{k!}}$$

w/ $\mu = n \cdot p$

N... # of events of interest in a certain time period (arrivals of job orders to a manufacturing facility, visits @ an ATM, fire alarms going off)

$$\boxed{N \sim \text{Poisson}(\lambda)}$$

the parameter: $\lambda \in (0, \infty)$

We say: N has the Poisson distribution w/ parameter λ

Support$(N) = \mathbb{N}_0$, i.e., N is $\mathbb{N}_0$-valued

The pmf of N is

$$\boxed{p_n = e^{-\lambda} \cdot \frac{\lambda^n}{n!} \quad \text{for all } n \in \mathbb{N}_0}$$

Problem. The # of children born in a single week in a hospital is modelled w/ a Poisson dist'n w/ parameter $\lambda = 20$. What's the probability that exactly 20 children were born in this hospital in a week?

$\rightarrow$: N... # of babies in a week

$N \sim \text{Poisson}(\lambda = 20)$

$$\mathbb{P}[N = 20] = e^{-20} \cdot \frac{20^{20}}{20!} = e^{-20} \cdot \frac{20^{19}}{19!}$$

$\square$

Q: Do we have expressions for the mean and the variance of N as functions of the parameter λ?

1^{st} We suspect, knowing the Poisson approximation to the binomial that

$$\text{mean} = \mu = np = \lambda$$

and

$$\text{var} = np\underbrace{(1-p)}_{\approx 1} \approx np = \mu = \lambda$$

2^{nd} We can justify the above guesses:

- $\mathbb{E}[N] = \sum_{n=0}^{+\infty} p_n \cdot n$

$$= \sum_{n=1}^{+\infty} \left(e^{-\lambda} \frac{\lambda^n}{\cancel{n!}_{(n-1)!}} \cdot \cancel{n} \right) = e^{-\lambda} \cdot \sum_{n=1}^{+\infty} \frac{\lambda^n}{(n-1)!} \underset{m=n-1}{=} e^{-\lambda} \cdot \boxed{\sum_{m=0}^{+\infty} \frac{\lambda^{m+1}}{m!}}$$

$$= \cancel{e^{-\lambda}} \cdot \lambda \cdot \cancel{e^{\lambda}} = \lambda$$

- $\text{Var}[N] = \mathbb{E}[N^2] - (\mathbb{E}[N])^2 = \lambda$

Look @ the text for the proof!

Problem. An actuary discovered that policyholders are three times as likely to file two claims as to file four claims.
If the number of claims filed has the Poisson dist'n, what is the variance of the number of claims filed?

$\longrightarrow$: $N \sim \text{Poisson}(\lambda)$

$$p_2 = 3 \cdot p_4$$

$$\cancel{e^{-\lambda}} \cdot \frac{\cancel{\lambda^2}}{\cancel{2!}} = \cancel{3} \cdot \cancel{e^{-\lambda}} \cdot \frac{\lambda^{\cancel{4}\,2}}{4!}$$

$$4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}$$

$\Rightarrow \lambda^2 = 4 \Rightarrow \lambda = 2 \Rightarrow$ answer: 2 $\square$

Example. N_1... # of customers w/ short questions
N_2... # of customers w/ long questions } w/in a specific time interval

Assume: $N_i \sim \text{Poisson}(\lambda_i)$, $i = 1, 2$ $\boxed{\text{independent}}$

N... the total # of customers in that time interval

$$\boxed{N = N_1 + N_2}$$

N is $\mathbb{N}_0$-valued

For $n \in \mathbb{N}_0$:

$$p_N(n) = \mathbb{P}[N = n] = \mathbb{P}[N_1 + N_2 = n]$$

independent

$$= \sum_{j=0}^{n} \left(p_{N_1}(j) \cdot p_{N_2}(n-j) \right)$$

$$= \sum_{j=0}^{n} \left(e^{-\lambda_1} \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \frac{\lambda_2^{n-j}}{(n-j)!} \right)$$

$$= e^{-\lambda_1} \cdot e^{-\lambda_2} \frac{1}{n!} \underbrace{\boxed{\sum_{j=0}^{n} \left(\frac{\cancel{1}\, n!}{j!(n-j)!} \cdot \lambda_1^j \cdot \lambda_2^{n-j} \right)}}_{(\lambda_1 + \lambda_2)^n}$$

$$\boxed{p_N(n) = e^{-(\lambda_1 + \lambda_2)} \cdot \frac{(\lambda_1 + \lambda_2)^n}{n!}}$$

$$\Rightarrow \boxed{N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2)}$$

$\square$

Problem. The number of thin-crust pizzas ordered from Harry's pizzeria in an afternoon is Poisson w/ mean 10.
The number of deep-dish pizzas is also Poisson w/ mean 5.
The orders of different types of pizzas are independent.

Q: What is the expected total # of pizzas ordered?

$\longrightarrow$: $\boxed{10 + 5 = 15}$

Doesn't use Poisson or independence!

Q: What is the probability that the total # of pizzas exceeds 2?

$\longrightarrow$: N... total # of pizzas

$N \sim \text{Poisson}(\lambda = 15)$

$$\mathbb{P}[N > 2] = 1 - \mathbb{P}[N=0] - \mathbb{P}[N=1] - \mathbb{P}[N=2]$$

$$= 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} - e^{-\lambda} \cdot \frac{\lambda^1}{1!} - e^{-\lambda} \cdot \frac{\lambda^2}{2!}$$

$$= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

$$= 1 - e^{-15}\left(1 + 15 + \frac{15^2}{2}\right)$$

$$= 1 - e^{-15}\left(1 + 15 + \frac{225}{2}\right)$$

$$= 1 - e^{-15}(16 + 112.5) = 1 - 128.5e^{-15}$$

$\square$