M362K: April 10^{th}, 2024.

More on Discrete Distributions.

Last HW Problem.

A_1, A_2, A_3 are 3 events $\boxed{\text{w/ indicator r.v. } \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3}$

w/ p_1, p_2, p_3

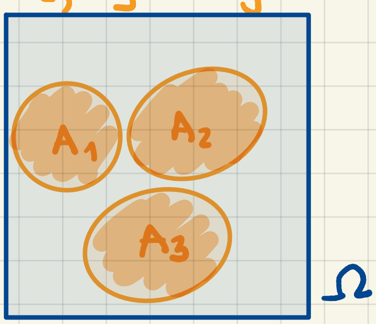
N... # of events that happen

a) $\boxed{N = \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3}$

b) $\mathbb{E}[N] = \mathbb{E}[\mathbb{I}_1] + \mathbb{E}[\mathbb{I}_2] + \mathbb{E}[\mathbb{I}_3] = p_1 + p_2 + p_3$

linearity

c) A_1, A_2, A_3 disjoint



$$N \sim \begin{cases} 1 & \text{w/ probab. } \boxed{p_1 + p_2 + p_3 = p} \\ 0 & \text{w/ probab. } 1 - p \end{cases}$$

$\mathrm{Var}[N] = \boxed{p(1-p)}$

Bernoulli

d) A_1, A_2, A_3 independent

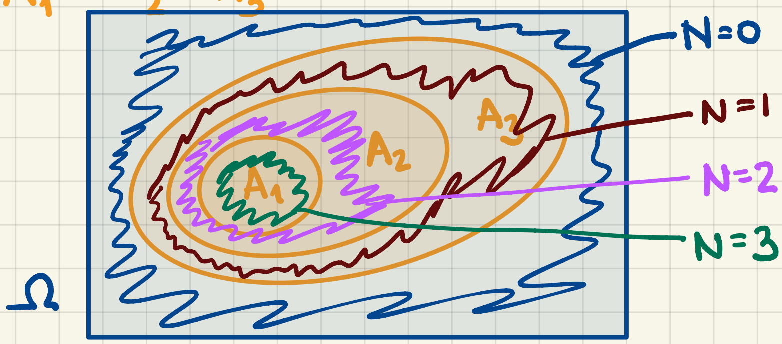
$\Downarrow$

$\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3$ independent

$\mathrm{Var}[N] = \mathrm{Var}[\mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3] = \mathrm{Var}[\mathbb{I}_1] + \mathrm{Var}[\mathbb{I}_2] + \mathrm{Var}[\mathbb{I}_3]$

$= p_1(1-p_1) + p_2(1-p_2) + p_3(1-p_3)$

e) $A_1 \subset A_2 \subset A_3$



To do: Find the pmf of N and calculate the variance using

$\mathrm{Var}[N] = \mathbb{E}[N^2] - (\mathbb{E}[N])^2$

Problem. An insurance company determines that N, the number of claims received in a week, is a r.v. w/

$$\boxed{p_n = \mathbb{P}[N=n] = \boxed{\frac{1}{2^{n+1}}}} \quad \text{where } n = 0, 1, \ldots$$

The company also determines that the number of claims received in a given week is independent of the number of claims in any other week.

Find the probability that exactly 7 claims will be received during a given two-week period.

→: N_i ... # of claims in week i, $i=1,2$

M ... total # of claims in the two weeks

$M = N_1 + N_2$

$$\boxed{\mathbb{P}[M=7] = ?}$$

$\text{Support}(M) = \mathbb{N}_0$

For $m \in \mathbb{N}_0$:

$$\mathbb{P}[M=m] = \mathbb{P}[N_1+N_2=m]$$

N_1 and N_2 independent

$$= \sum_{j=0}^{m} \left(p_{N_1}(j) \cdot p_{N_2}(m-j) \right)$$

$$= \sum_{j=0}^{m} \left(\frac{1}{2^{\cancel{j}+1}} \cdot \frac{1}{2^{m-\cancel{j}+1}} \right)$$

$$= \sum_{j=0}^{m} \frac{1}{2^{m+2}} = \boxed{\frac{m+1}{2^{m+2}}}$$

$$\mathbb{P}[M=7] = \frac{7+1}{2^{7+2}} = \frac{2^3}{2^9} = \frac{1}{2^6} = \frac{1}{64} \quad \square$$

Example. Let T_1 and T_2 be independent geometric r.v.s w/ success probabilities p_1 and p_2 respectively.

$$\boxed{T = \min(T_1, T_2)}$$

T_1's Bernoulli trials: F F F F F F S — 1st Kind

T_2's Bernoulli trials: F F F S — 2nd Kind

For this particular elementary outcome ω:

$$\left.\begin{matrix} T_1(\omega) = 7 \\ T_2(\omega) = 4 \end{matrix}\right\} T(\omega) = \min(T_1, T_2)(\omega) = 4$$

The experiment corresponding to T:

Repeat the Bernoulli trials w/ success probability p until the first success.

$p = \mathbb{P}$[@ least one of the 1st Kind and 2nd Kind of trials is a success]

$p = 1 - \mathbb{P}$[both are failures]

$$p = 1 - (1-p_1)(1-p_2) \overset{\text{independent}}{\Rightarrow} \boxed{T \sim g(p = 1-(1-p_1)(1-p_2))}$$

□

Example. "Mean Time to Failure"

A machine has the probability p of failing within every hour. We check whether the machine is still running every hour (on the hour).

Assume that failures in different hours happen independently.

How long (in hours) do we EXPECT to wait until the first failure?

Def'n. The expectation (or expected value or mean) of a discrete r.v. X is

$$\boxed{\mathbb{E}[X] = \sum_{x \in \text{Support}(X)} x \cdot p_X(x)}$$

if the series is absolutely convergent

In this example: $\boxed{N \sim g(p)}$

$$\mathbb{E}[N] = \sum_{n=1}^{\infty} \left(n \cdot p_N(n)\right) = \sum_{n=1}^{\infty} \left(n \cdot (1-p)^{n-1} \cdot p\right)$$

$$= p \sum_{n=1}^{\infty} n (1-p)^{n-1} = \frac{1}{p} \quad \square$$

Look @ the Tail-Sum Formula for the Expectation.

T is $\mathbb{N}$-valued

$$\mathbb{E}[T] = \sum_{n=1}^{\infty} n \cdot p_T(n)$$

$$= 1 \cdot p_T(1)$$

$$+ 1 \cdot p_T(2) + 1 \cdot p_T(2)$$

$$+ 1 \cdot p_T(3) + 1 \cdot p_T(3) + 1 \cdot p_T(3)$$

$$\vdots$$

$$+ 1 \cdot p_T(n) + 1 \cdot p_T(n) + 1 \cdot p_T(n) + \cdots + 1 \cdot p_T(n)$$

$$\vdots$$

$$= \mathbb{P}[T > 0] + \mathbb{P}[T > 1] + \mathbb{P}[T > 2] + \cdots + \mathbb{P}[T > n-1] + \cdots$$

For the geometric:

$$1 + q + q^2 + \cdots + q^{n-1} + \cdots$$

$$= \frac{1}{1-q} = \frac{1}{p}$$

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