# M362K: April 5th, 2024.

## Section 3.4. Discrete Distributions

So far: Usually, we focused on $\Omega$s that were finite and we had Xs that had finite supports.

Now: We want to model, e.g., $\boxed{\text{the time}}$ until some event of interest happens.

In some integer units.

We would have an outcome space such as:

$$\Omega = \underbrace{\{0, 1, 2, \dots, n, \dots\}}_{\text{An Infinite } \Omega!}$$

More generally: Want to talk about COUNTABLE outcome spaces, i.e., $\Omega$s whose elements can be arranged in a sequence, i.e.,

$$\Omega = \{\omega_1, \omega_2, \omega_3, \dots, \omega_n, \dots\}$$

More examples:

$$\Omega = \{1, 2, 3, \dots, n, \dots\}$$

$$\Omega = \{T, HT, HHT, \dots, H^{(n)}T, \dots\}$$

$$\Omega = \{0, 1, -1, 2, -2, 3, -3, \dots, n, -n, \dots\}$$

(for difference in times, toss a coin, ...)

Q: What was our first example of a probability dist'n on a finite $\Omega$?

$\longrightarrow$: EQUALLY LIKELY OUTCOMES w/ $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$ $\square$

Q: Is this type of a uniform distribution possible on an infinitely countable $\Omega$?

$\longrightarrow$: $\Omega = \{\omega_1, \omega_2, \dots, \omega_n, \dots\}$

Assume that there exists a probability $p > 0$ such that

$$\boxed{p = \mathbb{P}[\omega_i]} \quad \text{for all } i \in \mathbb{N}$$

By the Law of Total One,

$$\underline{1} = \sum_{i=1}^{+\infty} \boxed{\mathbb{P}[\omega_i]} = \sum_{i=1}^{+\infty} p = \text{"}\underline{p \cdot \infty}\text{"}$$

Uniform probabilities on infinitely countable $\Omega$ are impossible.

$\square$

**Example.** Consider a coin w/ probability of Hs equal to $p$.
Toss this coin repeatedly (and independently).
The first time the H appears (success!) is recorded.

$$\Omega = \mathbb{N} = \{1, 2, \dots, n, \dots\}$$

We can be forced to wait arbitrarily long.

What is the probability dist'n $\mathbb{P}$?

$$\mathbb{P}[\{1\}] = p$$
$$\mathbb{P}[\{2\}] = (1-p) \cdot p$$
$$\mathbb{P}[\{3\}] = (1-p)^2 \cdot p$$
$$\vdots$$
$$\mathbb{P}[\{n\}] = (1-p)^{n-1} \cdot p$$
$$\vdots$$

We might be interested in something like:

$\mathbb{P}[\{\text{it takes a prime number of trials}\}]$

**Def'n.** A function $T : \Omega \to \mathbb{R}$ is a random variable on $\Omega$.

e.g., $T$... a r.v. representing the number of trials until the first success (inclusive)

$$T(\omega) = \omega \quad \text{for all } \omega \in \Omega$$

$$\boxed{\text{Support}(T) = \mathbb{N} = \{1, 2, \dots\}}$$

**Def'n.** Any r.v. whose support is @ most countable (i.e., finite or infinitely countable) is called a DISCRETE RANDOM VARIABLE.

If Support$(T) = \mathbb{N}$, we say that $T$ is $\mathbb{N}$-valued.

If Support$(T) = \mathbb{N}_0 = \{0, 1, 2, \dots\}$, we say that $T$ is $\mathbb{N}_0$-valued.

**Def'n.** Let $T$ be a r.v. w/ support $\{t_1, t_2, \dots, t_n, \dots\}$. The function

$$p_T(t_i) = \mathbb{P}[T = t_i] \quad \text{for all } i \in \mathbb{N}$$

is the probability mass function of $T$.

**Convention.** If $T$ is $\mathbb{N}_0$-valued, then the pmf is usually written as

$$\boxed{p_k = p_T(k) \quad \text{for all } k \in \mathbb{N}_0}$$

**Example [cont'd].** $\boxed{T \sim g(p)}$

$$\boxed{p_n = (1-p)^{n-1} \cdot p \qquad \text{for all } n \in \mathbb{N}}$$

**Problem.** In modeling the number of claims filed by an individual under an auto policy during a 3-year period, an actuary makes a simplifying assumption that for all $n \geq 0$:

$$\boxed{p_{n+1} = \frac{1}{5} p_n}$$

w/ $p_n$ representing the probability that exactly $n$ claims were filed.

Under this assumption, what is the probability that a policyholder files more than one claim?

$\longrightarrow$:

$$p_{n+1} = \left(\tfrac{1}{5}\right) p_n = \left(\tfrac{1}{5}\right)\left(\tfrac{1}{5}\right) p_{n-1} = \cdots = \left(\tfrac{1}{5}\right)^{n+1} \cdot p_0$$

Similar to our geometric, but "shifted".

By the Law of Total One,

$$1 = \sum_{n=0}^{+\infty} p_n = \sum_{n=0}^{+\infty} \left(\left(\tfrac{1}{5}\right)^n p_0\right) = p_0 \boxed{\sum_{n=0}^{+\infty} \left(\tfrac{1}{5}\right)^n} = p_0 \boxed{\frac{1}{1-\left(\frac{1}{5}\right)}}$$

$$p_0 \cdot \frac{5}{4} = 1 \implies \boxed{p_0 = \frac{4}{5}} \checkmark$$

N... a r.v. denoting the # of claims

in the problem

$$p_1 = \frac{1}{5} p_0 = \frac{1}{5} \cdot \frac{4}{5} = \frac{4}{25}$$

$$\mathbb{P}[N > 1] = 1 - \mathbb{P}[N=0] - \boxed{\mathbb{P}[N=1]}$$

$$= 1 - \frac{4}{5} - \frac{4}{25} = \frac{25-20-4}{25} = \frac{1}{25} = 0.04$$

□

**Example.** Imagine that you toss a coin w/ probab. of Hs equal to $p$, until you get Heads.

Assume that you already tossed the coin $m$ times @ least not getting Hs. What's the probability that you have to toss the coin more than $n$ more times?

⟶: N... # of tosses

$$\boxed{N \sim g(p)}$$

Given that $\{N \geq m\} = \{N > m-1\}$

The probability we're looking for is

$$\mathbb{P}[N > n+m-1 \mid N > m-1] = ?$$

Hint: • Def'n of conditional probab.
• pmf of the geometric
• algebra

$$\mathbb{P}[N \geq n+m-1 \mid N > m-1] = \cancel{?} \frac{\mathbb{P}[N > n+m-1, N > m-1]}{\mathbb{P}[N > m-1]}$$

$$= \frac{\mathbb{P}[N > n+m-1]}{\mathbb{P}[N > m-1]} = \frac{(1-p)^{n+\cancel{m-1}}}{\cancel{(1-p)^{m-1}}} = (1-p)^n$$

$$= \mathbb{P}[N > n]$$

This is called the memoryless property.