# M362K: April 1st, 2024.

## Section 3.3. (cont'd).

**Review.** **Def'n.** For a discrete r.v. $X$ w/ pmf $p_X$, the expectation is defined as

$$\boxed{\mathbb{E}[X] = \sum_{\text{all } x} p_X(x) \cdot x}$$

if the sum exists.

**Fact.** Let $X_1, X_2, \dots, X_n$ be any r.v.s w/ finite expectations.
Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be constants.

Then,

$$\mathbb{E}[\alpha_1 X_1 + \alpha_2 X_2 + \dots + \alpha_n X_n] =$$

$$= \alpha_1 \mathbb{E}[X_1] + \alpha_2 \mathbb{E}[X_2] + \dots + \alpha_n \mathbb{E}[X_n]$$

linearity of $\mathbb{E}$

**Def'n.** For a random variable $X$ w/ a finite mean $\mu_X$, its variance is defined as

$$\boxed{\mathrm{Var}[X] = \mathbb{E}\left[(X-\mu_X)^2\right]}$$

Q: Is there an "addition rule" for the variance as well?

**Example.** $X$... # of successes in $n$ independent Bernoulli trials w/ success probab. $p$

$Y$... # of failures in the exact same trials

$$\mathrm{Var}[\underbrace{X+Y}_{n}] = ?$$

$$\mathrm{Var}[\;n\;] = 0$$

On the other hand, $\mathrm{Var}[X] > 0$ and $\mathrm{Var}[Y] > 0$

**Fact.** Let $X$ and $Y$ be two r.v.s on the same $\Omega$ and w/ finite variances.

$$\mathrm{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - (\mathbb{E}[X+Y])^2 = \dots$$

computational formula for Var

$$\dots = \mathbb{E}[X^2 + 2XY + Y^2] - (\overbrace{\mathbb{E}[X]}^{\mu_X} + \overbrace{\mathbb{E}[Y]}^{\mu_Y})^2$$

$$= \boxed{\mathbb{E}[X^2]} + 2\mathbb{E}[XY] + \boxed{\mathbb{E}[Y^2]}$$

$$- (\boxed{\mu_X^2} + 2\mu_X\mu_Y + \boxed{\mu_Y^2})$$

$$= \mathrm{Var}[X] + 2\underbrace{(\mathbb{E}[XY] - \mu_X\mu_Y)}_{\mathrm{Cov}[X,Y]} + \mathrm{Var}[Y]$$

We got the covariance formula:

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + 2\,\mathrm{Cov}[X,Y] + \mathrm{Var}[Y]$$

Example [cont'd].

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[X, n-X] = \mathbb{E}[X(n-X)] - \mathbb{E}[X]\cdot(n - \mathbb{E}[X])$$

$$= \cancel{n\mathbb{E}[X]} - \mathbb{E}[X^2] - \cancel{n\cdot\mathbb{E}[X]} + (\mathbb{E}[X])^2$$

$$= -\left(\mathbb{E}[X^2] - (\mathbb{E}[X])^2\right) = -\mathrm{Var}[X]$$

$$\mathrm{Var}[Y] = \mathrm{Var}[\overset{\text{shift}}{n} - X] = \mathrm{Var}[-X] = \mathrm{Var}[-1\cdot X] = (-1)^2\,\mathrm{Var}[X] = \mathrm{Var}[X]$$

By the covariance formula:

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + 2\,\mathrm{Cov}[X,Y] + \mathrm{Var}[Y]$$

$$= \mathrm{Var}[X] + 2\cdot(-\mathrm{Var}[X]) + \mathrm{Var}[X] = 0$$

**Proposition** If $X$ and $Y$ are independent, then

$$\boxed{\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]}$$

addition rule for the variances.

**Def'n.** The correlation (coefficient) between r.v.s $X$ and $Y$ is

$$\boxed{\rho_{X,Y} = \mathrm{corr}[X,Y] = \frac{\mathrm{Cov}[X,Y]}{\mathrm{SD}[X]\cdot\mathrm{SD}[Y]}}$$

**Def'n.** If $\rho_{X,Y} = 0$, then we say that $X$ and $Y$ are uncorrelated

In general,

$$\boxed{-1 \le \rho_{X,Y} \le 1}$$

Problem. The profit for a new product is given by

$$W = 3X - Y - 5.$$

Assume that X and Y are $\boxed{\text{independent}}$ w/ $Var[X]=1$ and $Var[Y]=2$.

What is the variance of W?

$\longrightarrow$:

$$Var[W] = Var[3X - Y \overset{\text{shift}}{-5}] = $$

$$= Var[3X - Y] \underset{\substack{\uparrow \\ \text{addition rule} \\ \text{for Var}}}{=} Var[3X] + \underbrace{Var[-Y]}_{= Var[(-1)\cdot Y] = (-1)^2 \cdot Var[Y]}$$

$$= 9 \cdot Var[X] + Var[Y] = 9 \cdot 1 + 2 = 11 \quad \square$$

Fact. Say that $X_1, X_2, \dots, X_n$ are independent r.v.s w/ finite variances.

Then,

$$\boxed{Var[X_1 + X_2 + \cdots + X_n] = Var[X_1] + Var[X_2] + \cdots + Var[X_n]}$$

Example. $X \sim Binomial(n = \#\text{of trials}, p = \text{success probability})$

Review. $\boxed{\mathbb{E}[X] = n \cdot p}$

We used $X = I_1 + I_2 + \cdots + I_n$

w/ $\boxed{I_j \sim Bernoulli(p)}$, $j = 1..n$, independent.

$$Var[X] = Var[I_1 + I_2 + \cdots + I_n]$$

$$= Var[I_1] + Var[I_2] + \cdots + Var[I_n]$$

identically distributed

$$= n \cdot \boxed{Var[I_1]}$$

$$Var[I_1] = \mathbb{E}[I_1^2] - (\mathbb{E}[I_1])^2 = \overset{=p}{\mathbb{E}[I_1]} - (\mathbb{E}[I_1])^2$$

$$Var[I_1] = p - p^2 = p(1-p) = \boxed{pq}$$

$$\boxed{Var[X] = n \cdot p(1-p)} \Rightarrow \boxed{SD[X] = \sqrt{n \cdot p(1-p)}}$$

All was well in deMoivre Laplace.

## Square Root Law.

Consider $\{X_1, X_2, \dots\}$, a sequence of independent, identically distributed (iid) random variables.

Let $S_n = X_1 + X_2 + \dots + X_n$

and

$$\overline{X}_n = \frac{1}{n} S_n = \frac{X_1 + \dots + X_n}{n}$$