University of Texas at Austin

Problem Set # 14

Power of Test.

Provide your **complete solution** for the following problems.

Problem 14.1. As the sample size increases, the power of a test will increase. *True or false? Why?*

→: As a proof of concept: The left-sided alternative.

$$H_0: \mu = \mu_0 \quad \text{vs.} \quad H_a: \mu < \mu_0$$

The RR is of the form: $\boxed{(-\infty, \boxed{?}]}$

$\boxed{?} = \mu_0 + z_\alpha\left(\frac{\sigma}{\sqrt{n}}\right)$ w/ $z_\alpha = \Phi^{-1}(\alpha) = qnorm(\alpha)$

Note: $z_\alpha < 0$

Let μ_a be a value from the alternative, i.e., $\boxed{\mu_a < \mu_0}$. Under that alternative, the sample mean has this dist'n:

$$\bar{X} \sim Normal\left(mean = \mu_a, sd = \frac{\sigma}{\sqrt{n}}\right)$$

The power of the test @ μ_a:

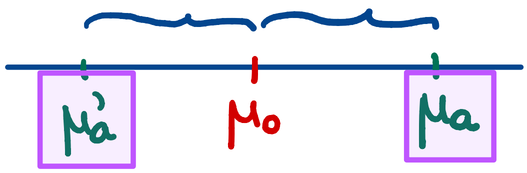
$$\mathbb{P}_{\mu_a}\left[\bar{X} \le \mu_0 + z_\alpha \boxed{\left(\boxed{\frac{\sigma}{\sqrt{n}}}\right)}\right]$$

$n \uparrow$ $\quad$ $\uparrow$ w/ n

□

Problem 14.2. (2 points) Consider a two-sided hypothesis test for the population mean of a normal population. Then, the power of the test is symmetric with respect to the null mean. *True or false? Why?*

$\longrightarrow$: For every μ_a from the alternative, we can "calculate" the power of the test @ that μ_a



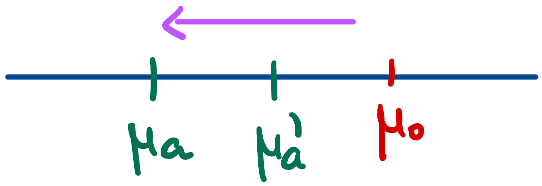
The claim is that the power @ μ_a' is equal.
Argue geometrically ☺

The power of the test can be understood as a <u>function</u> whose domain are all the values from the alternative.

Problem 14.3. (2 points) Let μ denote the population mean μ of a normally distributed population model with a known σ. At a given significance level α, we are testing

$$H_0 : \mu = \mu_0 \quad vs. \quad H_a : \mu < \mu_0.$$

Let μ_a and μ_a' be two values in the alternative such that $\mu_a < \mu_a'$. Then, the power of the test at the alternative μ_a exceeds the power of the test at the alternative μ_a'. *True or false? Why?*



Recall the work from Problem #1:

$$\mathbb{P}_{\mu_a}\left[\bar{X} \le \mu_0 + z_\alpha \cdot \left(\frac{\sigma}{\sqrt{n}}\right)\right] =$$

$$= \mathbb{P}\left[\overset{Z}{\frac{\bar{X} - \mu_a}{\frac{\sigma}{\sqrt{n}}}} \le \boxed{\frac{\mu_0 + z_\alpha \cdot \left(\frac{\sigma}{\sqrt{n}}\right) - \mu_a}{\frac{\sigma}{\sqrt{n}}}}\right]$$

$$= \Phi\left(z_\alpha + (\mu_0 - \mu_a) \cdot \boxed{\frac{\sqrt{n}}{\sigma}}\right) \quad \uparrow \text{ as } \mu_a \downarrow$$

Problem 14.4. The time needed for college students to complete a certain mirror-symmetry puzzle is modeled using a normal distribution with a mean of 30 seconds and a standard deviation of 3 seconds. You wish to see if the population mean time μ is changed by vigorous exercise, so you have a group of $\boxed{\text{nine}}$ college students exercise vigorously for 30 minutes and then complete the puzzle.

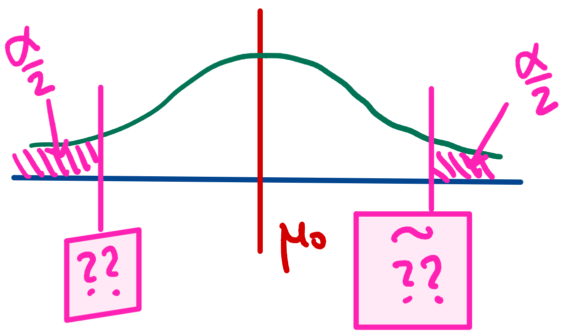
9

i. What are your null and alternative hypotheses?
ii. What is the rejection region at the significance level 0.01?
iii. What is the power of your test at $\mu = 28$ seconds?

i. $\underline{H_0 : \mu = 30}$ vs. $H_a: \mu \neq 30$

ii. $\alpha = 0.01$

The shape of the RR is:



$$\boxed{??} = \mu_0 + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) = 30 + (-2.576)\cdot\frac{3}{\sqrt{9}} = \underline{27.424}$$

$$z_{\alpha/2} = \Phi^{-1}(0.005) = \text{qnorm}(0.005) = -2.575829$$

$$\boxed{\widetilde{??}} = \mu_0 - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) = 30 + 2.576 = \underline{32.576}$$

$$RR = (\underset{0}{\cancel{-\infty}}, 27.424] \cup [32.576, +\infty)$$

iii. $\mu_a = 28$ power of test = ?

"fail·to·reject" region = $(27.424, 32.576)$

$\bar{X} \sim \text{Normal}(\text{mean} = 28, \text{sd} = 1)$

$\beta = \mathbb{P}_{\mu_a}[27.424 < \bar{X} < 32.576] = \text{pnorm}(32.576, 28, 1) - \text{pnorm}(27.424, 28, 1)$

$= 0.7005535$

$\beta = \text{pnorm}(30 - \text{qnorm}(0.005), 28, 1) - \text{pnorm}(30 + \text{qnorm}(0.005), 28, 1) = \underline{0.71}$

Power of test $1 - \beta \approx 0.29$

Problem 14.5. (10 points) You believe that the mean pancake consumption at the pancake jamboree is more than 16 per person. So, you decide to test your hypothesis. You model the pancake consumption as normally distributed with an unknown mean and with variance equal to 4. The plan is to collect the information on the number of pancakes consumed from a sample of 64 people. Since you want to have everything ready for the big day, you work out the rejection region right away and you get $(16.4375, \infty)$.

(i) (5 points) What is the significance level used to obtain the above rejection region?

Right-sided Alternative.

The lower bound of the RR

$$16.4375 = 16 + z^* \cdot \left(\frac{2}{\sqrt{64}}\right)$$

$$z^* = 4(16.4375 - 16) = 1.75$$

1 - pnorm (1.75) = 0.04005916

(ii) (5 points) What is the power of the above test at the alternative mean of 17?

1 - pnorm(16.4375, 17, 0.25) = 0.9877755