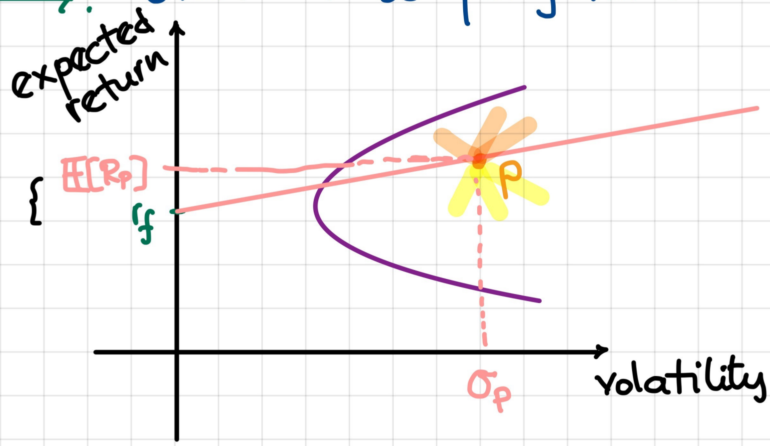
M339W: November 5th, 2021.

Required Returns.

Objective: To figure out if we can improve a portfolio by "adding" (more of) a particular security.

Q: What is the condition for the improvement of the portfolio and what consequences does this condition have on the desired expected return of the security?

→: Start w/ a portfolio P.



slope = Sharpe ratio of P

$$\boxed{\eta_P := \frac{\mathbb{E}[R_P] - r_f}{\sigma_P}}$$

Consider an investment I.

Construct P':

- keep P
- borrow $x \cdot$(Value of P) @ the rate r_f
- invest $x \cdot$(Value of P) in the investment I

Assume the weight x is small!

The new return: $R_{P'} = R_P - x \cdot r_f + x \cdot R_I$

⇒ The risk premium of P':

$$\mathbb{E}[R_{P'}] - r_f = \mathbb{E}[R_P] - x \cdot r_f + x \cdot \mathbb{E}[R_I] - r_f$$

$$= \underbrace{\left(\mathbb{E}[R_P] - r_f\right)}_{\text{the risk premium of P}} + \boxed{x \cdot \underbrace{\left(\mathbb{E}[R_I] - r_f\right)}_{\text{the risk premium of I}}} \quad ✓$$

The variance of $R_{P'}$:

$$\sqrt{Var[R_{P'}]} = Var[R_P - \underbrace{x\cdot r_f}_{\text{deterministic}} + x\cdot R_I] =$$

$$= Var[R_P + x\cdot R_I] =$$

$$= \sqrt{\underbrace{Var[R_P]}_{y_0} + \underbrace{2\cdot x\cdot Cov[R_P, R_I]}_{dy}} + \overbrace{\left(\underbrace{x^2\cdot Var[R_I]}_{\text{lower order term, i.e., diversified since } x \text{ is small}}\right)}^{\text{negligible}}$$

$$f(y) = \sqrt{y} = y^{1/2}$$

$$f'(y) = \frac{1}{2}\cdot y^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{y}}$$

$$f(y_0 + dy) = f(y_0) + f'(y_0)dy + \text{lower order terms}$$

$$= f(y_0) + \frac{1}{2}\cdot\frac{1}{\sqrt{y_0}}dy + \dots$$

$$\sqrt{Var[R_{P'}]} = \sqrt{Var[R_P]} + \frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{\sqrt{Var[R_P]}}}\cdot\cancel{2}\cdot x\cdot \overbrace{Cov[R_P, R_I]}^{corr[R_P,R_I]\cdot \cancel{SD[R_P]}\cdot SD[R_I]}$$

$$SD[R_{P'}] = SD[R_P] + \boxed{x\cdot SD[R_I]\cdot corr[R_P, R_I]}$$

"incremental" risk added to the portfolio by "adding" I.

Our criterion is:

$$\cancel{x}\left(\mathbb{E}[R_I] - r_f\right) > \underbrace{\cancel{x}\cdot SD[R_I]\cdot corr[R_P, R_I]\cdot \eta_P}_{\text{the effect of staying on the line through P w/ the same "incremental" risk added}}$$

$$\mathbb{E}[R_I] - r_f > SD[R_I] \cdot corr[R_P, R_I] \cdot \frac{\mathbb{E}[R_P] - r_f}{SD[R_P]}$$

$$\mathbb{E}[R_I] > r_f + \boxed{\frac{\sigma_I}{\sigma_P} \cdot \rho_{P,I}} (\mathbb{E}[R_P] - r_f)$$

$$\shortparallel$$

β_I^P ... the beta of the investment I w/ the portfolio P

<u>Def'n</u>. The required return of investment I given portfolio P:

$$\boxed{r_I := r_f + \beta_I^P (\mathbb{E}[R_P] - r_f)}$$

<u>Important Consequence</u>:

<u>Recall</u>: A portfolio P^* is efficient if no other portfolio outperforms it (in the sense of the Sharpe ratio).

Imagine there is an investment I such that

$$\mathbb{E}[R_I] > r_I = r_f + \beta_I^{P^*} (\mathbb{E}[R_P] - r_f)$$

$\Rightarrow$ Portfolio P^* can be improved by investing in I.

$\Rightarrow\Leftarrow$ Contradicts the fact that P^* is efficient.

$\Rightarrow$ For any security I:

$$\boxed{\mathbb{E}[R_I] = r_f + \beta_I^* (\mathbb{E}[R_{P^*}] - r_f)}$$

beta of investment I w/ the efficient portfolio P^*.