Exchange Options.

T... exercise date

two risky assets $\begin{cases} S \text{ ... underlying asset} \\ Q \text{ ... strike asset} \end{cases}$

Exchange call: $V_{EC}(T, S, Q) = (S(T) - Q(T))_+$

Exchange put: $V_{EP}(T, S, Q) = (Q(T) - S(T))_+$

$\Rightarrow$ a special symmetry: $V_{EC}(T, S, Q) = V_{EP}(T, Q, S)$

$\Rightarrow$ $\boxed{V_{EC}(0, S, Q) = V_{EP}(0, Q, S)}$

no arbitrage

It suffices to develop the Black-Scholes pricing formula for exchange calls.

Black-Scholes.

- S... underlying asset : δ_S ... dividend yield

  σ_S ... volatility

Our goal is pricing. So, we look @ S under the risk-neutral probability measure:

$$S(T) = S(0)e^{\left(r - \delta_S - \frac{\sigma_S^2}{2}\right)\cdot T + \sigma_S\sqrt{T}\cdot Z_S}$$

$Z_S \sim N(0,1)$

- Q ... strike asset : δ_Q... dividend yield

  σ_Q ... volatility

Under the risk-neutral probability measure:

$$Q(T) = Q(0)\, e^{\left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right)\cdot T + \sigma_Q\sqrt{T}\cdot Z_Q}$$

$Z_Q \sim N(0,1)$

w/ ρ... the correlation coefficient between Z_S and Z_Q.

Black-Scholes Price.

$$V_{EC}(0, S, Q) = F^P_{0,T}(S) \cdot N(d_1) - F^P_{0,T}(Q) \cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{F^P_{0,T}(S)}{F^P_{0,T}(Q)}\right) + \frac{1}{2}\sigma^2 \cdot T\right]$$

and $d_2 = d_1 - \sigma\sqrt{T}$

where $$\boxed{\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q}$$ ✓

Note: $\begin{cases} \bullet\ S(t),\ t \geq 0 \\ \bullet\ Q(t),\ t \geq 0 \end{cases}$

For every t:

$$Var\left[\ln\left(\frac{S(t)}{Q(t)}\right)\right] = Var\left[\ln(S(t)) - \ln(Q(t))\right]$$

under the risk-neutral measure

$$= Var\left[\boxed{\ln(S(0)) + \left(r - \delta_S - \frac{\sigma_S^2}{2}\right) \cdot t} + \sigma_S\sqrt{t} \cdot Z_S - \left(\boxed{\ln(Q(0)) + \left(r - \delta_Q - \frac{\sigma_Q^2}{2}\right) \cdot t} + \sigma_Q\sqrt{t} \cdot Z_Q\right)\right]$$

deterministic

$$= Var\left[\sigma_S \sqrt{t} Z_S - \sigma_Q \sqrt{t} Z_Q\right]$$

$$= t \cdot Var\left[\sigma_S \cdot Z_S - \sigma_Q \cdot Z_Q\right]$$

$$= t\left(\sigma_S^2 \cdot \underbrace{Var[Z_S]}_{1} + \sigma_Q^2 \cdot \underbrace{Var[Z_Q]}_{1} - 2\sigma_S \cdot \sigma_S \cdot \underbrace{Cov[Z_S, Z_Q]}_{=}\right)$$

$$\underbrace{\underbrace{corr[Z_S, Z_Q]}_{\rho} \cdot \underbrace{SD[Z_S]}_{= 1} \cdot \underbrace{SD[Z_Q]}_{= 1}}_{\rho}$$

$$= t\boxed{\left(\sigma_S^2 + \sigma_Q^2 - 2\sigma_S \cdot \sigma_Q \cdot \rho\right)}$$

We can rewrite the BS price as:

$$V_{EC}(0,S,Q) = S(0)e^{-\delta_S \cdot T} \cdot N(d_1) - Q(0)e^{-\delta_Q \cdot T} \cdot N(d_2)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)e^{-\delta_S \cdot T}}{Q(0)e^{-\delta_Q \cdot T}}\right) + \frac{1}{2}\sigma^2 \cdot T\right]$$

$$= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{Q(0)}\right) + \left(\underbrace{\delta_Q}_{r} - \underbrace{\delta_S}_{\delta} + \frac{\sigma^2}{2}\right) \cdot T\right]$$