M339W: March 25th, 2022.

Delta-Hedger's Profit over a "Small" Time Period.

Set up:

1st An agent writes an option @ time 0.

2nd Then, the agent Δ-hedges @ time 0 by trading in the shares of the underlying asset.

=> At time 0, their wealth is:

$$\boxed{w(S(0),0) = \boxed{-\underbrace{v(S(0),0)}_{\substack{\text{the time-0} \\ \text{value of} \\ \text{the option} \\ \text{they wrote}}} + \underbrace{\Delta(S(0),0)}_{\substack{\text{the time-0} \\ \text{delta of} \\ \text{the written} \\ \text{option}}} \cdot S(0)}}$$

Consider the value of their portfolio @ time dt (before the portfolio is rebalanced to re-establish the Δ-hedge).

$$\boxed{w(S(dt), dt) = -v(S(dt), dt) + \Delta(S(0),0) \cdot S(dt)}$$

For a "small" time dt, we can use the Δ-Γ-Θ approximation:

$ds = S(dt) - S(0)$

$$\begin{aligned} w(S(dt), dt) \approx -\Big(& v(S(0), 0) + \cancel{\Delta(S(0), 0)\, ds} \\ & + \frac{1}{2}\Gamma(S(0),0)\,(ds)^2 \\ & + \Theta(S(0),0)\, dt \Big) \\ & + \Delta(S(0),0)\underbrace{S(dt)}_{=\cancel{ds}+S(0)} \end{aligned}$$

$$\begin{aligned} = & \boxed{-v(S(0),0) + \Delta(S(0),0) \cdot S(0)} \\ & - \frac{1}{2}\Gamma(S(0),0)\,(ds)^2 \quad ✓ \\ & - \Theta(S(0),0)\, dt \quad ✓ \end{aligned}$$

To calculate the approximate profit:

$$\boxed{\begin{aligned} & w(S(dt), dt) - w(S(0), 0)\, e^{r\,dt} \approx \\ & \approx w(S(0), 0)\left(1 - e^{r\,dt}\right) - \frac{1}{2}\Gamma(S(0), 0)\,(dS)^2 - \Theta(S(0), 0)\,dt \end{aligned}}$$

Delta-Gamma Hedging.

- Let's start w/ a Δ-neutral portfolio.
We denote the value f'tion of this portfolio by $v(s,t)$.
Then, we maintain $\boxed{\Delta(s,t) = 0}$ by trading in the shares of the underlying asset.
- The investor decides to establish a Γ-hedge as well, i.e., they want to maintain a Γ-neutral portfolio.

Q: Can they accomplish this by trading in the shares of the underlying asset alone?

$\longrightarrow$: Remember : the Γ of a stock investment is zero.

$\Rightarrow$ NO!

We need another option on the same underlying w/ a non-zero gamma
Denote the value f'tion of this option by $\tilde{v}(s,t)$;
its delta is $\tilde{\Delta}(s,t)$ and its gamma is $\tilde{\Gamma}(s,t)$.

Let $\tilde{N}(s,t)$ be the # of the "new" options necessary @ time t w/ stock price s in order to maintain Γ-neutrality.
The formal condition:

$$\Gamma(s,t) + \tilde{N}(s,t) \cdot \tilde{\Gamma}(s,t) = 0$$

$$\Rightarrow \boxed{\tilde{N}(s,t) = -\frac{\Gamma(s,t)}{\tilde{\Gamma}(s,t)}}$$

Now, the delta of the entire portfolio equals:

$$\boxed{\underbrace{\Delta(s,t)}_{=0}} + \tilde{N}(s,t) \cdot \tilde{\Delta}(s,t)$$

The Δ-neutrality can be reestablished just trading in the shares of stock. Let $N_S(s,t)$ be the # of shares necessary to accomplish that:

$$\tilde{N}(s,t)\cdot\tilde{\Delta}(s,t)+\boxed{N_S(s,t)}=0$$

Δ-neutrality

$$\boxed{N_S(s,t)=-\tilde{N}(s,t)\cdot\tilde{\Delta}(s,t)}$$

Recall, again, the Γ of the stock investment is zero.

So, Γ-neutrality remains. ∎

Task: Solve Problem #10 from MFE, Spring 2007.