M339W: March 21st, 2022.

17. Assume the Black-Scholes framework. Consider a one-year at-the-money European put option on a nondividend-paying stock.

$\boxed{T=1}$ $\boxed{S(0)=K}$ $\boxed{\delta=0}$

You are given:

(i) The ratio of the put option price to the stock price is less than 5%.

$$\boxed{\frac{v_p(S(0),0)}{S(0)} < 0.05}$$

(ii) Delta of the put option is –0.4364.

(iii) The continuously compounded risk-free interest rate is 1.2%. $\boxed{r=0.012}$

Determine the stock's volatility.

$\sigma = ?$

(A) 12%

(B) 14%

(C) 16%

(D) 18%

(E) 20%

$$\text{(ii)} \Rightarrow \Delta_p(S(0),0) = -0.4364$$

$$+\cancel{e^{-\delta\cdot T}}\cdot \boxed{N(-d_1(S(0),0)) = +0.4364}$$

$$N(d_1(S(0),0)) = 1-0.4364 = 0.5636$$

$$\boxed{d_1(S(0),0) = 0.16}$$

$$\frac{1}{\sigma\sqrt{1}}\left[\cancel{\ln\left(\frac{S(0)}{K}\right)} + \left(0.012+\frac{\sigma^2}{2}\right)\cdot 1\right] = 0.16$$

at-the-money

$$0.5\sigma^2 - 0.16\sigma + 0.012 = 0 \quad /\cdot 2$$

$$\boxed{\sigma^2 - 0.32\sigma + 0.024 = 0}$$

$$\Rightarrow \boxed{\sigma_1 = 0.12 \quad \text{and} \quad \sigma_2 = 0.20}$$

$$\text{(i)} \Rightarrow \cancel{K}e^{-rT}\cdot N(-d_2(S(0),0)) - \cancel{S(0)}\cdot \underbrace{N(-d_1(S(0),0))}_{0.4364} < 0.05\cdot \cancel{S(0)}$$

$$e^{-0.012}\cdot N(-d_2(S(0),0)) < 0.05 + 0.4364 = 0.4864$$

$$N(-d_2(S(0),0)) < e^{0.012}\cdot 0.4864 = 0.4923$$

By def'n: $d_2 = d_1 - \sigma\sqrt{T} \Rightarrow d_2(S(0),0) = 0.16-\sigma$

Choose: $\boxed{\sigma = 0.12}$

Example. Consider:

- an at-the-money $\boxed{S(0)=K}$ call/put w/ $r=\delta$ ✓

or

- a call/put on a non-dividend-paying stock w/ $K = S(0)e^{rT}$ ✓

or

- any choice of parameter values such that

$$\boxed{F^P_{0,T}(S) = PV_{0,T}(K)}$$

$$d_1(S(0),0) = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$= \frac{1}{\sigma\sqrt{T}}\left[\cancel{\ln\left(\frac{S(0)e^{-\delta T}}{Ke^{-rT}}\right)} + \frac{\sigma^2}{2}\cdot T\right] = \frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

The call price:

$$v_c(S(0),0) = S(0)e^{-\delta T}\cdot N(d_1(S(0),0)) - Ke^{-rT}\cdot N(d_2(S(0),0))$$

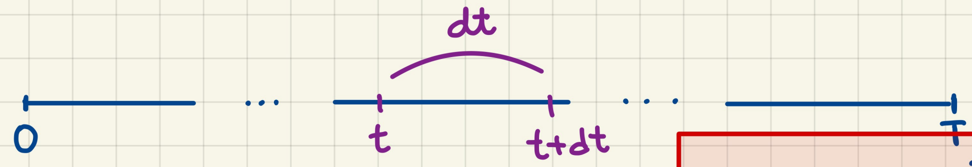
$$= F^P_{0,T}(S)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

$$= F^P_{0,T}(S)\left(2\cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

Given the price of the call, we can invert the price function to get the implied volatility. □

Delta · Gamma · Theta Approximation.

For any portfolio in our market model, we consider its value function $\boxed{v(s,t)}$



$Z \sim N(0,1)$

$$S(t) \qquad S(t+dt) = S(t)\,e^{\left(r-\delta-\frac{\sigma^2}{2}\right)\cdot dt + \sigma\sqrt{dt}\cdot Z}$$

$$s \qquad s+ds$$

$$\boxed{ds = S(t+dt) - S(t)}$$

$$v(s,t) \longrightarrow v(s+ds,\, t+dt)$$

Taylor-like expansion.

$$v(s+ds,\, t+dt) \cong v(s,t) + \underbrace{\frac{\partial}{\partial s} v(s,t)}_{=\Delta(s,t)} ds + \frac{1}{2}\underbrace{\frac{\partial^2}{\partial s^2} v(s,t)}_{=\Gamma(s,t)} (ds)^2 + \underbrace{\frac{\partial}{\partial t} v(s,t)}_{=\Theta(s,t)} dt$$

Delta · Gamma · Theta Approximation.

$\delta = 0$

19. Assume that the Black-Scholes framework holds. The price of a nondividend-paying stock is \$30.00. The price of a put option on this stock is \$4.00.

$S(0) = 30$ $\qquad v_p(S(0), 0) = 4$

You are given:

(i) $\Delta = -0.28$

(ii) $\Gamma = 0.10$

Using the delta-gamma approximation, determine the price of the put option if the stock price changes to \$31.50.

$S(dt) = 31.50$

(A) \$3.40

(B) \$3.50

(C) \$3.60

(D) \$3.70

(E) \$3.80

$$v_p(S(dt), dt) \approx v_p(S(0), 0) + \Delta_p(S(0), 0)\,ds + \frac{1}{2}\Gamma_p(S(0), 0)(ds)^2 \quad \text{w/ } ds = 31.50 - 30 = 1.50$$

****END OF EXAMINATION****

$$\text{answer} = 4 + (-0.28)(1.50) + \frac{1}{2}(0.10)(1.5)^2 = 3.69$$

20. Assume that the Black-Scholes framework holds. Consider an option on a stock.

You are given the following information at time 0:

(i) The stock price is $S(0)$, which is greater than 80. $S(0) > 80$

(ii) The option price is 2.34. $\boxed{v(S(0),0) = 2.34}$

(iii) The option delta is -0.181. $\boxed{\Delta(S(0),0) = -0.181}$

(iv) The option gamma is 0.035. $\boxed{\Gamma(S(0),0) = 0.035}$

$S(dt) = 86.$

The stock price changes to 86.00. Using the delta-gamma approximation, you find that the option price changes to 2.21.

$\underline{v(S(dt), dt) = 2.21.}$

Determine $S(0)$.

By the $\Delta \cdot \Gamma$ approximation:

$$v(S(dt), dt) = v(S(0),0) + \Delta(S(0),0)\,ds + \frac{1}{2}\Gamma(S(0),0)\,(ds)^2$$

(A) 84.80 : $ds = 1.20$

(B) 85.00 : $ds = 1$

(C) 85.20 : $ds = 0.80$

(D) 85.40 : $ds = 0.60$

(E) 85.80 : $ds = 0.20$

$$2.21 = 2.34 + (-0.181)\cdot ds + \frac{1}{2}(0.035)(ds)^2$$

Solve the quadratic.

****END OF EXAMINATION****