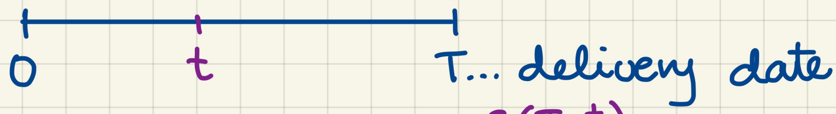
M339W: March 4th, 2022.

Option Greeks [cont'd].

Example. Prepaid Forward Contract on a Continuous Dividend Paying Stock.



$$F^P_{t,T}(S) = S(t)\,e^{-\delta(T-t)}$$

$$\Rightarrow \quad v(s,t,r,\delta,\sigma) = s e^{-\delta(T-t)}$$

$$\Rightarrow \quad \boxed{\Delta(\cdots) = \frac{\partial}{\partial s} v(\cdots)} = e^{-\delta(T-t)}$$

$$\Gamma(\cdots) = \frac{\partial^2}{\partial s^2} v(\cdots) = 0$$

$$\Theta(\cdots) = \frac{\partial}{\partial t} v(\cdots) = \delta \cdot s e^{-\delta(T-t)} > 0$$

$$\psi(\cdots) = \frac{\partial}{\partial \delta} v(\cdots) = -(T-t)\, s\, e^{-\delta(T-t)} < 0$$

Focus on the Delta.

value f'tion : $v(s,t,r,\delta,\sigma) = v(s,t)$

the Delta : $\boxed{\Delta(s,t) := \frac{\partial}{\partial s} v(s,t)}$ ✓

European call.

K... strike price
T... exercise date

Black-Scholes

$$v_C(s,t) = \boxed{s e^{-\delta(T-t)} \cdot N(d_1(s,t))} - K e^{-r(T-t)} \cdot N(d_2(s,t))$$

w/ $$d_1(s,t) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)(T-t)\right]$$

and $d_2(s,t) = d_1(s,t) - \sigma\sqrt{T-t}$

By def'n of the Delta:

$$\Delta_c(s,t) = \boxed{\frac{\partial}{\partial s} \boxed{v_c(s,t)}}$$

After the chain rule and the product rule:

$$\boxed{\Delta_c(s,t) = e^{-\delta(T-t)} \cdot N(d_1(s,t))} > 0$$

The positivity of the call Δ makes sense due to the fact that the call is a long position w.r.t. the underlying.

European Put.

$$\boxed{\begin{array}{l} K \ldots \text{ strike price} \\ T \ldots \text{ exercise date} \end{array}}$$

Put-Call Parity.

$$\frac{\partial}{\partial s} \Big| \quad v_c(s,t) - \boxed{v_p(s,t)} = \boxed{s e^{-\delta(T-t)}} - K e^{-r(T-t)}$$

$$\Delta_c(s,t) - \Delta_p(s,t) = e^{-\delta(T-t)}$$

$$\Delta_p(s,t) = \Delta_c(s,t) - e^{-\delta(T-t)} =$$

$$= e^{-\delta(T-t)} N(d_1(s,t)) - e^{-\delta(T-t)}$$

$$= e^{-\delta(T-t)} \big(N(d_1(s,t)) - 1\big)$$

$$\boxed{\Delta_p(s,t) = - e^{-\delta(T-t)} \big(N(-d_1(s,t))\big)} < 0$$

The negativity makes sense since the put is short w.r.t. the underlying.

(A) 7.32 million

(B) 7.42 million

(C) 7.52 million

(D) 7.62 million

(E) 7.72 million

$T = 1/4$ $K = 41.5$

8. You are considering the purchase of a 3-month 41.5-strike American call option on a nondividend-paying stock.

→ Never optimal to exercise early. ⇒ Equivalent to a European call.

You are given: $\boxed{\delta = 0}$

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 40. $S(0)=40$

(iii) The stock's volatility is 30%. $\boxed{\sigma = 0.3}$

(iv) The current call option delta is 0.5. $\boxed{\Delta_C(S(0),0) = 0.5}$

Determine the current price of the option. $\boxed{v_c(S(0),0) = ?}$

No dividends!

$\boxed{N(d_1(S(0),0)) = 0.5}$

(A) $20 - 20.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(B) $20 - 16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(C) $20 - 40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx$

(D) $16.138\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

(E) $40.453\int_{-\infty}^{0.15} e^{-x^2/2}dx - 20.453$

$$v_c(S(0),0) = \underbrace{\boxed{S(0)}}_{40}\, e^{-\delta \cdot T} \underbrace{\boxed{N(d_1(S(0),0))}}_{0.5} - Ke^{-rT}\cdot \boxed{N(d_2(S(0),0))}$$

$= 1$

$\boxed{r = ?}$ $\boxed{d_1(S(0),0) = 0}$

$$\Rightarrow \frac{1}{\sigma\sqrt{T}}\underbrace{\left[\ln\left(\frac{40}{41.5}\right) + \left(r + \frac{0.09}{2}\right)\cdot\frac{1}{4}\right]}_{=0} = 0$$

$$r + 0.045 = 4\ln\left(\frac{41.5}{40}\right) \Rightarrow \boxed{r = 0.10226} \checkmark$$

$$\boxed{d_2(S(0),0) = ?}$$

$$d_2(S(0),0) = d_1(S(0),0) - \sigma\sqrt{T} = 0 - 0.3\sqrt{\frac{1}{4}} = -0.3\left(\frac{1}{2}\right) = -0.15$$

$$v_c(S(0),0) = 40\,(0.5) - 41.5\,e^{-0.10226(0.25)} \cdot N(-0.15)$$

$$= 20 - 40.4525 \cdot N(-0.15)$$

$$= 20 - 40.4525\,(1 - N(0.15))$$

symmetry of $N(0,1)$

$$= 40.4525\,N(0.15) + 20 - 40.4525$$

$$= \underbrace{40.4525\,\underbrace{N(0.15)}_{\int_{-\infty}^{0.15} f_Z(z)\,dz} \quad \int_{-\infty}^{0.15} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz}_{\boxed{\frac{40.4525}{\sqrt{2\pi}}} \int_{-\infty}^{0.15} e^{-\frac{z^2}{2}}\,dz} - 20.4525$$

$$\frac{40.4525}{\sqrt{2\pi}} = 16.138 \Rightarrow (D)$$

■