M339W: March 2nd, 2022.

Option Greeks.

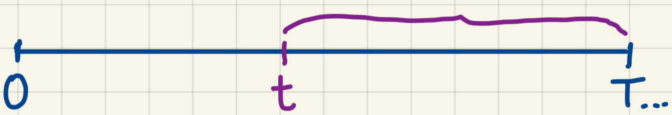
Objective. To study the dependence of the value of our portfolio on these independent arguments:

$$s, \quad t, \quad r, \quad \delta, \quad \sigma$$

s: the asset price @ a particular time t

t: valuation date

$(T-t)$ ... time to exercise



T ... time horizon (say, liquidation date or exercise date)

We use the Black-Scholes model for the stock price.

$S(t)$, $t \geq 0$ ... time-t stock price

Under the risk-neutral probability measure $\mathbb{P}^*$:

$$S(T) = S(t)e^{(r-\delta-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

Consider the B-S price @ time-t of a European call/put w/ strike K and exercise date T.

$$V_C(t) = \underbrace{S(t)\cdot e^{-\delta(T-t)}}_{\text{prepaid forward price}} \cdot N(d_1) - \underbrace{Ke^{-r(T-t)}}_{\text{present value of strike}} \cdot N(d_2)$$

and

$$V_P(t) = Ke^{-r(T-t)}\cdot N(-d_2) - S(t)e^{-\delta(T-t)}\cdot N(-d_1)$$

w/ $$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S(t)}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)\right]$$

and $$d_2 = d_1 - \sigma\sqrt{T-t}$$

We (re)introduce: s...an independent argument which stands for the time t ("current") stock price

In our pricing formula:

$$v_C(s,t,r,\delta,\sigma) = se^{-\delta(T-t)} \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2)$$

and

$$v_P(s,t,r,\delta,\sigma) = Ke^{-r(T-t)} N(-d_2) - se^{-\delta(T-t)} \cdot N(-d_1)$$

w/ $d_1^{(\dots)} = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{s}{K}\right) + \left(r-\delta+\frac{\sigma^2}{2}\right)(T-t)\right]$

and $d_2^{(\dots)} = d_1 - \sigma\sqrt{T-t}$

In this course: We consider portfolios consisting of:

- the riskless asset
- the risky asset (think: continuous-dividend-paying stock)
- European options on the risky asset

$\Rightarrow$ We are able to represent the value of such a portfolio as a value function, i.e.,

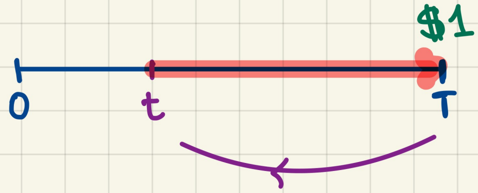
$$\boxed{v(s,t,r,\delta,\sigma)}$$

Def'n. Greeks.

- $\frac{\partial}{\partial s} v(\dots) =: \Delta(\dots)$ DELTA
- $\frac{\partial^2}{\partial s^2} v(\dots) =: \Gamma(\dots)$ GAMMA
- $\frac{\partial}{\partial t} v(\dots) =: \Theta(\dots)$ THETA

- $\frac{\partial}{\partial r} v(\dots) =: \rho(\dots)$ RHO
- $\frac{\partial}{\partial \delta} v(\dots) =: \psi(\dots)$ PSI
- $\frac{\partial}{\partial \sigma} v(\dots) =: vega(\dots)$ VEGA

Example. Consider a zero-coupon bond redeemable @ time T for \$1.
This is the only component of your portfolio.
Then, the value f'tion is:



$$v(s,t,r,\delta,\sigma) = \underline{e^{-r(T-t)}}$$

$$\Delta(\cdots) = 0$$

$$\Gamma(\cdots) = 0$$

$$\Theta(\cdots) = \frac{\partial}{\partial t}\left(e^{-r(T-t)}\right) = re^{-r(T-t)} > 0$$

$$\rho(\cdots) = \frac{\partial}{\partial r}\left(e^{-r(T-t)}\right) = -(T-t)e^{-r(T-t)} < 0$$

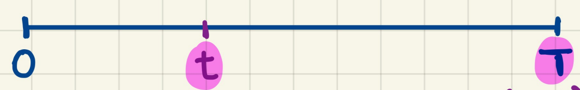
Example. Consider a portfolio consisting of one share of non-dividend-paying stock.

$\Rightarrow$ Its value f'tion: $v(s,t,r,\delta,\sigma) = s$

$$\boxed{\begin{aligned}\Delta(\cdots) &= 1\\ \Gamma(\cdots) &= 0\end{aligned}}$$

Example. Prepaid forward contract on a continuous-dividend-paying stock.

T... delivery date



$$F^P_{t,T}(S) = S(t)\cdot e^{-\delta(T-t)}$$

Think about Greeks!

$\Rightarrow$ The value function is:

$$v(s,t,r,\delta,\sigma) = se^{-\delta(T-t)}$$