M339W: February 23rd, 2022.

Black·Scholes Pricing.

- In general, under a probability measure $\mathbb{P}$:

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/} \quad Z \sim N(0,1)$$

w/ α... mean rate of return

- Under the risk·neutral probability measure $\mathbb{P}^*$:

$$S(T) = S(0)e^{(r - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z^*} \quad \text{w/} \quad Z^* \sim N(0,1)$$

w/ r ... continuously compounded, risk·free interest rate

Under a probability measure $\mathbb{P}$:

$$\boxed{\mathbb{E}[V_c(T)] = S(0)e^{(\alpha-\delta)T}\cdot N(\hat{d}_1) - K\cdot N(\hat{d}_2)}$$

w/ $\hat{d}_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (\alpha - \delta + \frac{\sigma^2}{2})\cdot T\right]$

and $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}$

By the risk·neutral pricing principle:

$$\boxed{V(0) = e^{-rT}\mathbb{E}^*[\underbrace{V(T)}_{\text{the payoff of a European option}}]}$$

Call Options.

$$V_c(0) = e^{-rT}\left(S(0)e^{(r-\delta)T}\cdot N(d_1) - K\cdot N(d_2)\right)$$

$$= \cancel{e^{-rT}}\cdot S(0)e^{(\cancel{r}-\delta)T}\cdot N(d_1) - Ke^{-rT}N(d_2)$$

$$= \underbrace{\boxed{S(0)e^{-\delta T}}}_{=F^P_{0,T}(S)} N(d_1) - \underbrace{\boxed{Ke^{-rT}}}_{=PV_{0,T}(K)} N(d_2)$$

Black-Scholes Call Price:

$$V_c(0) = F^P_{0,T}(S)\, N(d_1) - PV_{0,T}(K) \cdot N(d_2)$$

w/

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\cdot T\right]$$

and $d_2 = d_1 - \sigma\sqrt{T}$

$K=25$

6. You are considering the purchase of 100 units of a 3-month 25-strike European call option on a stock.

$\boxed{T=1/4}$ ✓

You are given:

(i) The Black-Scholes framework holds.

(ii) The stock is currently selling for 20. $S(0)=20$ ✓

(iii) The stock's volatility is 24%. $\boxed{\sigma=0.24}$

(iv) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%. $\delta=0.03$

(v) The continuously compounded risk-free interest rate is 5%. $r=0.05$

Calculate the price of the block of 100 options.

The Usual Steps.

$1^{\underline{st}}$ Find d_1 and d_2.

$2^{\underline{nd}}$ Use software/tables to get $N(d_1)$ and $N(d_2)$.

$3^{\underline{rd}}$ Calculate the BS Price

(A) 0.04

(B) 1.93

(C) 3.63

(D) 4.22

(E) 5.09

7. Company A is a U.S. international company, and Company B is a Japanese local company. Company A is negotiating with Company B to sell its operation in Tokyo to Company B. The deal will be settled in Japanese yen. To avoid a loss at the time when the deal is closed due to a sudden devaluation of yen relative to dollar, Company A has decided to buy at-the-money dollar-denominated yen put of the European type to hedge this risk.

You are given the following information:

(i) The deal will be closed 3 months from now.

(ii) The sale price of the Tokyo operation has been settled at 120 billion Japanese yen.

(iii) The continuously compounded risk-free interest rate in the U.S. is 3.5%.

(iv) The continuously compounded risk-free interest rate in Japan is 1.5%.

(v) The current exchange rate is 1 U.S. dollar = 120 Japanese yen.

(vi) The daily volatility of the yen per dollar exchange rate is 0.261712%.

(vii) 1 year = 365 days; 3 months = ¼ year.

Calculate Company A's option cost.

$\longrightarrow$:

1st $\quad d_1 = \dfrac{1}{\sigma\sqrt{T}}\left[\ln\left(\dfrac{S(0)}{K}\right) + \left(r-\delta+\dfrac{\sigma^2}{2}\right)\cdot T\right]$

$$d_1 = \frac{1}{0.24\sqrt{\frac{1}{4}}}\left[\ln\left(\frac{20}{25}\right) + \left(\underline{0.05} - 0.03 + \frac{(0.24)^2}{2}\right)\cdot\frac{1}{4}\right]$$

$$d_1 = -1.7579$$

and $\quad d_2 = -1.7579 - 0.24\left(\frac{1}{2}\right) = -1.8779$

2nd $\quad N(d_1) = \text{pnorm}(-1.7579) = 0.03938226$

$\quad N(d_2) = \text{pnorm}(-1.8779) = 0.03019742$

3rd

$$V_c(0) = S(0)e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2)$$

$$= 20e^{-0.03(1/4)}\cdot 0.0394 - 25e^{-0.05(1/4)}\cdot 0.0302$$

$$\cong 0.0365$$

$\Longrightarrow$ Our answer : 3.65 ■

$\delta = 0$

8. Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T, $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate.

$K = S(0)e^{rT}$

You are given:

(i) $\boxed{S(0) = \$100}$

(ii) $\boxed{T = 10}$

(iii) $\text{Var}[\ln S(t)] = 0.4t, \quad t > 0$.

$$S(t) = S(0)e^{\mu t + \sigma\sqrt{t}\,Z}$$

$$\ln(S(t)) = \ln(S(0)) + \mu t + \sigma\sqrt{t}\cdot Z$$

Determine the price of the call option.

$$\text{Var}[\ln(S(t))] = \text{Var}[\sigma\sqrt{t}\cdot Z]$$

$$= \sigma^2 \cdot t \cdot \text{Var}[Z]$$

$$= \sigma^2 \cdot t \underset{\substack{\uparrow \\ \text{(iii)}}}{=} 0.4t$$

$$\boxed{\sigma = \sqrt{0.4}}$$ ✓

(A) $7.96

(B) $24.82

(C) $68.26

(D) $95.44

(E) There is not enough information to solve the problem.

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{\cancel{S(0)}}{\cancel{S(0)}e^{rT}}\right) + \left(r + \frac{\sigma^2}{2}\right)\cdot T\right]$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\cancel{-rT} + \cancel{rT} + \frac{\sigma^2 \cdot T}{2}\right] = \frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{\sigma\sqrt{T}}{2} - \sigma\sqrt{T} = -\frac{\sigma\sqrt{T}}{2}$$

$$V_c(0) = S(0)\,N\left(\frac{\sigma\sqrt{T}}{2}\right) - (S(0)\cancel{e^{rT}})\cancel{e^{-rT}}\cdot N\left(-\frac{\sigma\sqrt{T}}{2}\right)$$

$$V_c(0) = S(0)\left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right) \underset{\uparrow}{=} S(0)\left(2\cdot N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1\right)$$

symmetry of $N(0,1)$

$$V_c(0) = 100\,(2N(1) - 1)$$
$$= 68.26895$$