Tail Probabilities.

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Example. You are considering an investment in a continuous dividend paying stock.
You want to compare it to the risk-free investment.

Q: What is the probability that the stock outperforms the risk-free account @ time T?

→: The invested amount: $S(0)$

- If it's the risk-free investment, the balance @ time T is:

$$S(0) \cdot e^{rT}$$

- If it's the stock investment, the number of shares owned @ time T is:

$$e^{\delta \cdot T}$$

⇒ The wealth @ time T is: $S(T) \cdot e^{\delta \cdot T}$

$$\boxed{\mathbb{P}[\underbrace{e^{\delta \cdot T} S(T) > S(0) e^{rT}}] = ?}$$

This question is equivalent to the question of whether the profit from purchase of stock is positive.

In the Black-Scholes model:

$$\boxed{S(T) = S(0) e^{(\alpha - \delta - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} \quad \text{w/} \quad Z \sim N(0,1)}$$

$$\mathbb{P}[\cancel{e^{\delta T}} \cdot \cancel{S(0)} e^{(\alpha - \cancel{\delta} - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} > \cancel{S(0)} e^{rT}] =$$

$$= \mathbb{P}[e^{(\alpha - \frac{\sigma^2}{2}) \cdot T + \sigma\sqrt{T} \cdot Z} > e^{rT}]$$

($\log(\cdot)$ increasing)

$$= \mathbb{P}[(\alpha - \frac{\sigma^2}{2}) T + \sigma\sqrt{T} \cdot Z > rT]$$

$$= \mathbb{P}[\sigma\sqrt{T} \cdot Z > (r - \alpha + \frac{\sigma^2}{2}) T]$$

$$= \mathbb{P}[Z > \left(\frac{r - \alpha + \frac{\sigma^2}{2}}{\sigma}\right)\sqrt{T}]$$

(symmetry of $N(0,1)$)

$$= \mathbb{P}[Z < -\left(\frac{r - \alpha + \frac{\sigma^2}{2}}{\sigma}\right)\sqrt{T}]$$

(N... cdf of $N(0,1)$)

$$= N\left(\left(\frac{\alpha - r - \frac{\sigma^2}{2}}{\sigma}\right)\sqrt{T}\right) \quad ✓$$

Q: What if we look @ the above example under the risk-neutral probability measure $\mathbb{P}^*$?

$\longrightarrow$: Under $\mathbb{P}^*$, we have $\boxed{\alpha = r.}$

So, we get

$$\mathbb{P}^*\left[e^{\delta \cdot T} \cdot S(T) > S(0)e^{rT}\right] = N\left(-\frac{\frac{\sigma^{\cancel{2}}}{2}}{\cancel{\sigma}}\sqrt{T}\right)$$

$$= N\left(-\frac{\sigma\sqrt{T}}{2}\right) \blacksquare$$

Motivation. Consider a European call option w/ strike price K and exercise date T. What is the probability that this option is in-the-money on its exercise date?

$\longrightarrow$: In our Black-Scholes model:

$$S(T) = S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} \quad \text{w/ } Z \sim N(0,1)$$

We are calculating:

$$\mathbb{P}[S(T) > K] =$$

$$= \mathbb{P}\left[S(0)e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > K\right]$$

$$= \mathbb{P}\left[e^{(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z} > \frac{K}{S(0)}\right] \quad (\ln(\cdot) \text{ increas.})$$

$$= \mathbb{P}\left[(\alpha - \delta - \frac{\sigma^2}{2})\cdot T + \sigma\sqrt{T}\cdot Z > \ln\left(\frac{K}{S(0)}\right)\right]$$

$$= \mathbb{P}\left[\boxed{\sigma\sqrt{T}}\cdot Z > \ln\left(\frac{K}{S(0)}\right) - (\alpha - \delta - \frac{\sigma^2}{2})\cdot T\right]$$

$$= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S(0)}\right) - (\alpha - \delta - \frac{\sigma^2}{2})\cdot T\right]\right]$$

(symmetry of N(0,1))

$$= \mathbb{P}\left[Z < -\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{K}{S(0)}\right) - (\alpha - \delta - \frac{\sigma^2}{2})\cdot T\right]\right]$$

$$= \mathbb{P}\left[Z < \boxed{\frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + (\alpha - \delta - \frac{\sigma^2}{2})\cdot T\right]}\right]$$

Introduce:

$$\boxed{\hat{d}_2 := \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S(0)}{K}\right) + \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T\right]}$$

shorthand!

$$\boxed{\mathbb{P}[S(T) > K] = N(\hat{d}_2)} \quad \checkmark$$

Consequently: The probability that the otherwise identical put is in-the-money @ time T is:

$$\mathbb{P}[S(T) < K] = 1 - \mathbb{P}[S(T) \geq K]$$
$$= 1 - N(\hat{d}_2) = N(-\hat{d}_2)$$

$$\boxed{\mathbb{P}[S(T) < K] = N(-\hat{d}_2)}$$

Problem. Let the current stock price be \$100.
Assume the Black-Scholes model.
You're given:

(i) • $\mathbb{P}[S(1/4) < 95] = 0.2358$

(ii) • $\mathbb{P}[S(1/2) < 110] = 0.6026$

What's the expected time-1 stock price?

$\longrightarrow$: $\boxed{\mathbb{E}[S(T)] = S(0)e^{(\alpha-\delta)\cdot T}}$

In the B-S model:

$$S(T) = S(0)e^{\overbrace{(\alpha-\delta-\frac{\sigma^2}{2})}^{=:\mu}\cdot T + \sigma\sqrt{T}Z} \quad \checkmark$$

In particular:

$$\mathbb{E}[S(1)] = S(0)e^{(\alpha-\delta)} = S(0)e^{\mu+\frac{\sigma^2}{2}}$$

(i): $\begin{cases} 95 \text{ is the } 23.58^{th} \text{ quantile of } S(1/4) \\ \text{The } 23.58^{th} \text{ quantile of } N(0,1): \ qnorm(0.2358) = -0.72 \end{cases}$

$$95 = 100\, e^{\mu (1/4) + \sigma \sqrt{1/4} \cdot (-0.72)}$$