M339W: February 11th, 2022.

Log-Normal Stock Prices.

Temporarily fix a time-horizon T.

$S(T)$... time-T stock price

$R(0,T)$... realized return over $(0,T)$

$$R(0,T) = \ln\left(\frac{S(T)}{S(0)}\right) \iff \boxed{S(T) = S(0)e^{R(0,T)}}$$

Recall: $R(0,T) \sim Normal(mean = m, variance = \nu^2)$

$\Rightarrow$ $S(T)$ is lognormal, and

$$\boxed{\mathbb{E}[S(T)] = S(0)e^{m+\frac{\nu^2}{2}}} \quad ☆$$

Market Model.

- RISKLESS ASSET w/ the ccrfir r
- RISKY ASSET: for now, a continuous-dividend-paying stock
  - δ... dividend yield ✓
  - $\boxed{\sigma\text{... volatility}}$
  - α... (mean) rate of return, i.e., ✓

    the constant which satisfies:

$$\boxed{S(0)e^{\alpha \cdot T} = e^{\delta \cdot T}\mathbb{E}[S(T)]}$$

i.e., $$\boxed{\mathbb{E}[S(T)] = S(0)e^{(\alpha-\delta)\cdot T}} \quad ☆☆ \quad ✓$$

Equating ☆ & ☆☆, we get

$$m + \frac{\nu^2}{2} = (\alpha-\delta)\cdot T$$

$$\boxed{m = (\alpha-\delta)\cdot T - \frac{\nu^2}{2}}$$

Consider $\mathrm{Var}[R(0,T)] = ?$?

Recall: $\mathrm{Var}[R(0,1)] = \underline{\;\sigma^2\;}$

$SD[R(0,1)] = \sigma$

by def'n

$\mathrm{Var}[R(0,T)] = \underline{\sigma^2 \cdot T}$

Note: $\boxed{SD[R(0,T)] = \sigma\sqrt{T}}$

Finally: $\boxed{R(0,T) \sim \mathrm{Normal}\left(\text{mean} = \underline{(\alpha - \delta - \tfrac{1}{2}\sigma^2)T}, \text{var} = \underline{\sigma^2 T}\right)}$

Say that $Z \sim N(0,1)$. Then, we can express $R(0,T)$ as

$$\boxed{R(0,T) = \left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}$$

Hence: $\boxed{S(T) = S(0)\, e^{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma\sqrt{T}\cdot Z}}$ ⊛

Q: What's the median of $S(T)$?

$\longrightarrow$: $\boxed{S(0)e^{\left(\alpha - \delta - \frac{\sigma^2}{2}\right)\cdot T}}$ ✓

Note: $$\frac{\text{mean}}{\text{median}} = \frac{\cancel{S(0)}\, e^{\cancel{(\alpha-\delta)\cdot T}}}{\cancel{S(0)}\, e^{(\cancel{\alpha-\delta}-\frac{\sigma^2}{2})\cdot T}} = e^{\frac{\sigma^2 \cdot T}{2}}$$

On Pricing.

When our focus is the quality of an investment, say, in terms of an expected payoff of an option on our stock, we use the subjective probability, i.e., we use the parameter α.

When we're pricing, we do so under the risk-neutral measure.

Q: What do we adjust in our model ⊛ so that we are looking @ the stock price under the risk-neutral probability measure and we are able to price?

Recall: Consider a one-period binomial tree:

$$S(0)\ \begin{cases} \xrightarrow{\ p^*\ } S_u = u\cdot S(0) \\[2ex] \xrightarrow{\ 1-p^*\ } S_d = d\cdot S(0) \end{cases}$$

$$\underbrace{\qquad\qquad}_{h=T}$$

$$p^* = \frac{e^{(r-\delta)\cdot h} - d}{u-d}$$

Q: You invest in one share of continuous-dividend-paying stock @ time-0. What is the expected wealth @ time-T under the risk-neutral probability?

Task: Solve Problem Set #5.