M339W: February 9th, 2022.

Moment Generating Functions.

For any random variable Y,
and for independent arguments denoted by t,
we define the moment generating function (mgf) of Y
as the following function of t:

$$\boxed{M_Y(t) := \mathbb{E}\left[e^{Y\cdot t}\right]}$$

for all t such that the expectation exists, i.e., if it's finite

Note: • $M_Y(0) = \underline{\ 1\ }$ $\Rightarrow$ @ least $t=0$ is in the domain of M_Y

Goal: To understand e^X where $X \sim Normal(mean = m, var = \nu^2)$

Recall: Any normal X can be written as

$$\boxed{X = m + \nu \cdot Z}$$ w/ $Z \sim N(0,1)$

In general: Let a and b be constants.
Define: $\tilde{Y} = a + bY$

Then, $M_{\tilde{Y}}(t) = \mathbb{E}\left[e^{\tilde{Y}\cdot t}\right] = \mathbb{E}\left[e^{(a+bY)t}\right]$

$$= \mathbb{E}\left[e^{at} \cdot e^{bt\cdot Y}\right]$$

$$= e^{at} \cdot \mathbb{E}\left[e^{bt\cdot Y}\right] = e^{at} \cdot M_Y(bt)$$

In particular: Recall: $\boxed{M_Z(t) = e^{\frac{t^2}{2}}}$

For X as above:

$$M_X(t) = e^{mt} \cdot M_Z(\nu t) = e^{mt} \cdot e^{\frac{\nu^2 t^2}{2}} = e^{mt + \frac{\nu^2 t^2}{2}}$$ ✓

The LogNormal Distribution.

Def'n. Let $X \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$.
Define $\boxed{Y = e^X}$

We say that Y is **lognormally distributed**.

Q: What is the mean?

$\rightarrow$: $\mathbb{E}[Y] = \mathbb{E}[e^X] = \mathbb{E}[e^{X \cdot 1}] = M_X(1) = e^{m + \frac{\nu^2}{2}}$

Compare to: $\mathbb{E}[X] = m$.

Caveat: $\boxed{\mathbb{E}[e^X] \geq e^{\mathbb{E}[X]}}$

This is a special case of **Jensen's Inequality**.

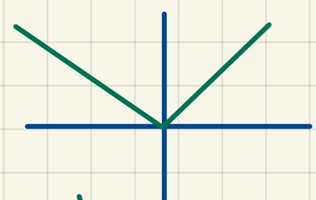
Theorem. Let X be a random variable, and g be a convex function such that

$g(X)$ is well-defined
and
$\mathbb{E}[g(X)]$ exists.

Then, $\boxed{\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])}$

Examples.

i. $g(x) = |x|$



$\Rightarrow \mathbb{E}[|X|] \geq |\mathbb{E}[X]|$

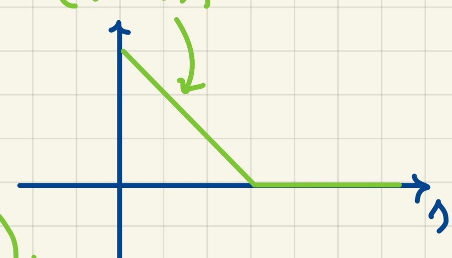
ii. Look @ a European put w/ strike K.

Its payoff function: $v(s) = (K - s)_+$

The expected payoff is:

$\mathbb{E}[(K - S(T))_+]$ $\geq$

Its lower bound: $(K - \mathbb{E}[S(T)])_+$

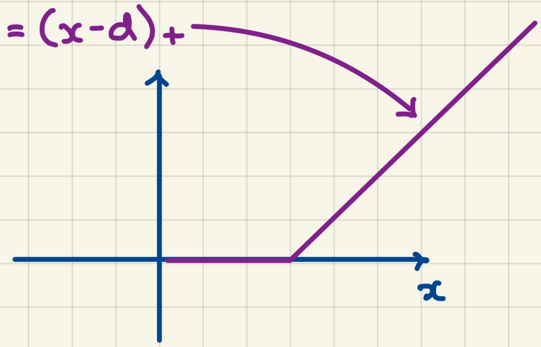


iii. In <u>classical insurance</u>:

$$\begin{cases} X \ldots \text{(ground-up) loss, i.e., severity r.v.} \\ d \ldots \text{deductible} \end{cases}$$

The insurer pays $(X-d)_+$

$\Rightarrow \quad g(x) = (x-d)_+$



<u>By Jensen's inequality</u>:

$$\boxed{\mathbb{E}\left[(X-d)_+\right] \geq \left(\mathbb{E}[X]-d\right)_+}$$

<u>Task</u>: Review <u>quantiles</u> and, in particular, the <u>median</u>!