M339W: February 7th, 2022.

Realized Returns.

With an agent's subjective probabilistic model for the return of a stock (or, equivalently, the stock price), we consider the following:

Temporarily fix a time $\cdot T$ (of some importance; say, you will want to assess your wealth then).

Say, you invest in $\boxed{\text{one}}$ share of a continuous $\cdot$ dividend $\cdot$ paying stock @ time $\cdot 0$. Let the dividend yield be denoted by δ.

Let $S(t), t \geq 0$ denote the time $\cdot t$ stock price. In particular, @ time $\cdot T$, the stock price is $S(T)$.

Q: What is your wealth @ time $\cdot T$?

$\longrightarrow$: $e^{\delta T} \cdot S(T)$

$\Rightarrow$ Your expected wealth @ time $\cdot T$ is: $e^{\delta \cdot T} \mathbb{E}[S(T)]$

0: $-S(0)$ ——— T: $\boxed{e^{\delta T} \mathbb{E}[S(T)]}$

Def'n. The mean rate of return is usually denoted by α and defined as the constant satisfying:

$$\boxed{S(0)e^{\alpha \cdot T} = e^{\delta T} \mathbb{E}[S(T)]} \checkmark$$

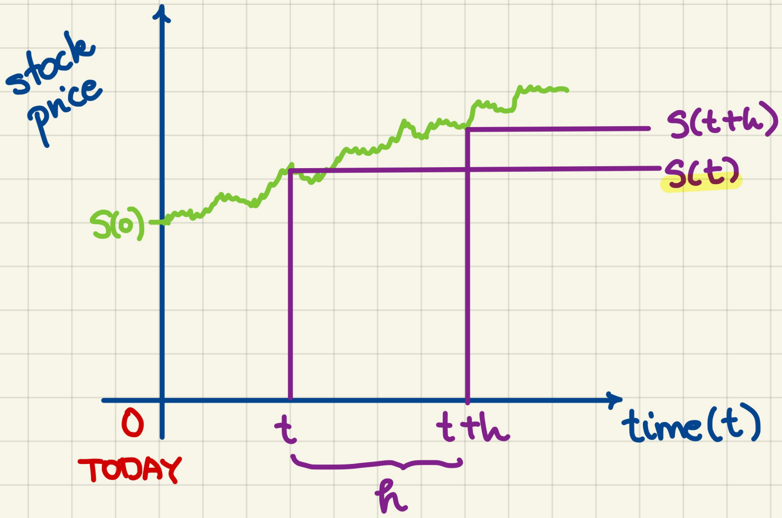
Note:
- We assume that α is a constant independent of the time horizon T.
- $\mathbb{E}[S(T)] = S(0) e^{(\alpha - \delta) \cdot T}$ ← mean rate of appreciation

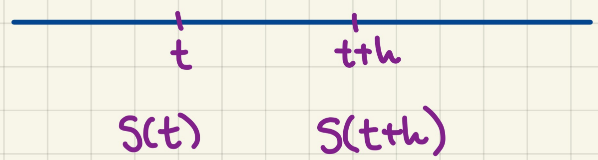
mean rate of return

α

$\delta + (\alpha - \delta)$

dividend yield ... mean rate of appreciation





Def'n. For every $t, h > 0$, we define the **realized return** $R(t, t+h)$ so that it satisfies

$$\boxed{\begin{gathered} S(t+h) = S(t)e^{R(t,t+h)} \\ \Longleftrightarrow \\ R(t,t+h) = \ln\left(\frac{S(t+h)}{S(t)}\right) \end{gathered}}$$

Recall: The volatility σ was defined as the standard deviation of realized return over any time period of length one year.

For instance, for $[0,1]$: $SD[R(0,1)] = \sigma$

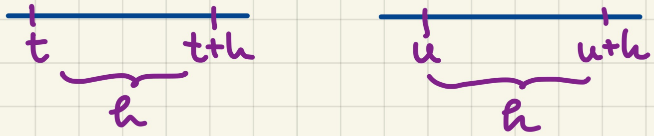
$$\Rightarrow Var[R(0,1)] = \sigma^2$$

Goal: To propose a model for $R(t,t+h)$, for $t, h > 0$, which has the following "properties":

- It inherits the desirable properties of the binomial tree, e.g. independent returns over disjoint time periods and time homogeneity.
- It can be interpreted as a limiting model of a binomial tree w/ $n \to \infty$.
- Its parametrization is in terms of α, δ, σ.

The properties of realized returns.

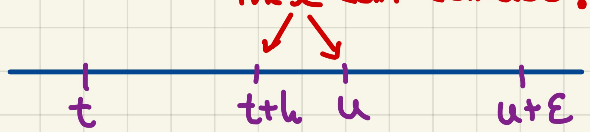
i. Time homogeneity.



We require that $R(t, t+h)$ and $R(u, u+h)$ to be identically distributed

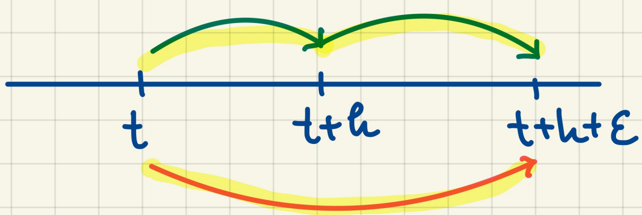
ii. Independence.

These can coincide!



We require that $R(t, t+h)$ and $R(u, u+\varepsilon)$ be independent.

iii. Additivity. Take $t, h, \varepsilon > 0$



By definition:

$$
\begin{aligned}
R(t, t+h+\varepsilon) &= \ln\left(\frac{S(t+h+\varepsilon)}{S(t)}\right)\\
&= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)} \cdot \frac{S(t+h)}{S(t)}\right)\\
&= \ln\left(\frac{S(t+h+\varepsilon)}{S(t+h)}\right) + \ln\left(\frac{S(t+h)}{S(t)}\right)\\
&= R(t+h, t+h+\varepsilon) + R(t, t+h)
\end{aligned}
$$

$$\boxed{R(t, t+h+\varepsilon) = R(t, t+h) + R(t+h, t+h+\varepsilon)}$$

We decide that realized returns will be modeled as normal, i.e.,

$R(t, t+h) \sim \text{Normal}(\text{mean} = m, \text{variance} = \nu^2)$