M339 J: March 31st, 2023.

A Statistics Note

Non Parametric

Parametric

↳ Focus on the distributions w/

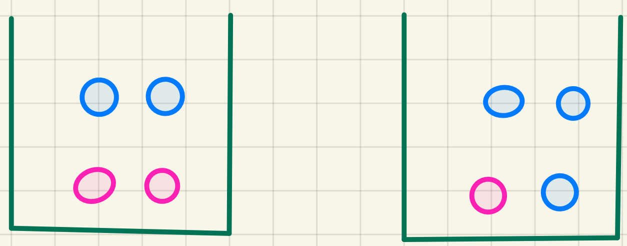
- cdf, pmf, pdf ... involving a parameter θ
- named dist'ns ... in terms of "a" parameter θ

<u>Caveat</u>: Sometimes your parameter is a vector.
Say, in the lognormal case:

$$\theta = (\underset{\in \mathbb{R}}{\mu}, \underset{>0}{\sigma})$$

Maximum likelihood Estimation.

Motivation.



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Maximum likelihood for Individual, Unmodified Data.

Let X_j, $j=1,\dots,n$, be continuous random variables. independent

Denote by f_{X_j} the pdf of X_j for $j=1,\dots,n$.

All the f_{X_j} must depend on the same parameter θ.

For now, our data set consists of singletons, i.e.,

$$x_1, x_2, \dots, x_n$$

The likelihood f'tion:

$$\boxed{L(\theta) = \prod_{j=1}^{n} f_{X_j}(x_j;\theta)}$$

Beware:

$$\mathbb{P}[X_j = x_j] = 0$$

$$\mathbb{P}[X_j \in dx_j] = f_{X_j}(x_j;\theta)\,dx_j$$

Introduce: $\ell(\theta) = \ln(L(\theta)) = \sum_{j=1}^{n} \ln(f_{X_j}(x_j;\theta))$

Now, we maximise the log-likelihood f'tion across θ.
Typically, this entails differentiation.

Example. $X_j \sim \text{Exponential}(\text{mean}=\theta)$, $j=1,\dots,n$

Data set: $x_1, x_2, \dots, x_n$

For every $j=1,\dots,n$, the pdf is

$$f_{X_j}(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, \quad x>0$$

The likelihood f'tion is:

$$L(\theta) = \prod_{j=1}^{n}\left(\frac{1}{\theta}e^{-\frac{x_j}{\theta}}\right) = \left(\frac{1}{\theta}\right)^n \prod_{j=1}^{n} e^{-\frac{x_j}{\theta}}$$

$$L(\theta) = \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum_{j=1}^{n}x_j}$$

The log-likelihood is:

$$\ell(\theta) = \ln(L(\theta)) = -n\ln(\theta) - \frac{1}{\theta}\sum_{j=1}^{n} x_j$$

We seek the maximum. So, we differentiate w.r.t. θ:

$$\ell'(\theta) = -n\cdot\frac{1}{\theta} + (+1)\cdot\frac{1}{\theta^2}\sum_{j=1}^{n} x_j = 0$$

$$\underbrace{\frac{1}{\theta^2}}_{\neq 0}\underbrace{\left(-n\cdot\theta + \sum_{j=1}^{n} x_j\right)}_{=0} = 0$$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{1}{n}\sum_{j=1}^{n} x_j = \bar{x}}$$

$\square$

Example. $X_j \sim Pareto(\alpha, \theta)$, $j = 1, \dots, n$

Data set: $x_1, x_2, \dots, x_n$

We assume that θ is known, and we're estimating α.

The likelihood f'tion:

$$L(\alpha, \theta) = \prod_{j=1}^{n} f(x_j; \alpha, \theta)$$

$$= \prod_{j=1}^{n} \frac{\alpha\cdot\theta^{\alpha}}{\boxed{(x_j+\theta)}^{\alpha+1}}$$

Q: What if we're given α and estimating θ?

$$L(\alpha, \theta) = \frac{\alpha^n\cdot\theta^{n\cdot\alpha}}{[(x_1+\theta)(x_2+\theta)\cdots(x_n+\theta)]^{\alpha+1}}$$

The log-likelihood f'tion:

$$\ell(\alpha, \theta) = n\cdot\ln(\alpha) + n\cdot\alpha\cdot\ln(\theta) - (\alpha+1)\sum_{j=1}^{n}\ln(x_j+\theta)$$

Omit θ from the notation and differentiate w.r.t. α:

$$l'(\alpha) = n \cdot \frac{1}{\alpha} + n \cdot \ln(\theta) - \sum_{j=1}^{n} \ln(x_j + \theta) = 0$$

$$\frac{n}{\alpha} = \sum_{j=1}^{n} \ln(x_j + \theta) - n \cdot \ln(\theta)$$

$$\boxed{\hat{\alpha}_{MLE} = \frac{n}{\sum_{j=1}^{n} \ln(x_j + \theta) - n \cdot \ln(\theta)}}$$