M339J: March 1st, 2023.

Poisson Thinning.

Thm. Let $N \sim \text{Poisson}(\lambda)$ be a counting random variable for some events of interest.
Suppose that independently from N, each of these events fall into a particular category indexed by $i = 1, 2, \dots, m$ w/ probability p_i $(i = 1 \dots m)$. $(p_1 + p_2 + \dots + p_m = 1)$

Let N_i be the count of events from category i, $i = 1, 2, \dots, m$.

Then,

- $N_i \sim \text{Poisson}(\lambda_i = p_i \cdot \lambda)$ for all $i = 1, \dots, m$
- $N_1, N_2, \dots, N_m$ are independent random variables.

111. The number of accidents follows a Poisson distribution with mean 12. Each accident generates 1, 2, or 3 claimants with probabilities 1/2, 1/3, and 1/6, respectively.

p_1 p_2 p_3

Calculate the variance of the total number of claimants.

S... the total # of claimants

$Var[S] = ?$

- N_i... # of accidents w/ i claimants, $i=1,2,3$

$N_i \sim Poisson(\lambda_i = \lambda \cdot p_i)$

$N_1 \sim Poisson(\lambda_1 = \underline{6})$

$N_2 \sim Poisson(\lambda_2 = \underline{4})$

$N_3 \sim Poisson(\lambda_3 = \underline{2})$

(A) 20

(B) 25

(C) 30

(D) 35

(E) 40

112. In a clinic, physicians volunteer their time on a daily basis to provide care to those who are not eligible to obtain care otherwise. The number of physicians who volunteer in any day is uniformly distributed on the integers 1 through 5. The number of patients that can be served by a given physician has a Poisson distribution with mean 30.

Determine the probability that 120 or more patients can be served in a day at the clinic, using the normal approximation with continuity correction.

(A) $1-\Phi(0.68)$

(B) $1-\Phi(0.72)$

(C) $1-\Phi(0.93)$

(D) $1-\Phi(3.13)$

(E) $1-\Phi(3.16)$

$$\underline{S = 1 \cdot N_1 + 2 \cdot N_2 + 3 \cdot N_3}$$

$$Var[S] = Var[N_1 + 2N_2 + 3N_3] =$$

N_1, N_2, N_3 are independent by the "thinning" Thm

$$= Var[N_1] + 4 Var[N_2] + 9 Var[N_3]$$
$$= \lambda_1 + 4\lambda_2 + 9\lambda_3$$
$$= 6 + 4(4) + 9(2) = 40 \quad \square$$

Binomial Coefficients.

For $n, k \in \mathbb{N}_0$ w/ $n \geq k$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can generalize to $x \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ such that $\boxed{x > k-1}$

$$\binom{x}{k} = \frac{\Gamma(x+1)}{\Gamma(k+1)\,\Gamma(x-k+1)}$$

Negative Binomial Distribution.

Inspiration : A Modelling Problem.

Consider a sequence of independent, identically dist'd Bernoulli trials.

Say : the probab. of success in every trial is p

The repetition of the Bernoulli trials continues until a total of r successes is achieved.

N... # of failures before the r^{th} success

Q : What's the support of N? $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

For $k = 0, 1, 2, \ldots$

The last trial must be a success!

$$p_N(k) = \underbrace{\mathbb{P}[N=k]}_{\substack{\text{What is the probability} \\ \text{of seeing } k \text{ failures} \\ \text{before the } r^{th} \text{ success?}}} = \binom{r+k-1}{k} p^r (1-p)^k$$

In this class, and the FAM-S Tables, we have this parameterization:

$$\boxed{p = \frac{1}{1+\beta}} \quad \text{for some } \beta > 0$$

With this parameterization:

$$p_N(k) = \binom{r+k-1}{k} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$$

Returning to our generalization of binomial coefficients, we can have

$$\boxed{r > 0, \ \beta > 0}$$

We write: $N \sim NegBinomial(r, \beta)$

- $\mathbb{E}[N] = r \cdot \beta$
- $Var[N] = r \cdot \beta(1+\beta) > \mathbb{E}[N]$
- $\boxed{P_N(z) = (1 - \beta(z-1))^{-r}}$

Q: What do we get in the special case where $r = 1$?

$\longrightarrow$: $\boxed{N \sim Geometric(\beta)}$

The geometric dist'n has the memoryless property.