Hw Problem: #3.2.

M339J: 01/30/23

$f_X(x) = \kappa x^{-5}$ for $x > 1$

$\kappa > 0$

$\boxed{\text{95\% percentile} = ?}$

$\pi_{0.95} = ?$ $\boxed{F_X(\pi_{0.95}) = 0.95}$

$\rightarrow$: $\kappa = ?$

$$\kappa \cdot \int_1^{+\infty} x^{-5}\,dx = 1$$

$$\kappa \cdot \left(\frac{1}{-5+1} x^{-5+1}\right)_{x=1}^{+\infty} = 1$$

$$\kappa \cdot (-0.25)(x^{-4})_{x=1}^{\infty} = 1$$

$$\kappa \cdot (-0.25)(0-1) = 1 \quad \Rightarrow \quad \boxed{\kappa = 4}$$

$\boxed{F_X(x) = ?}$

$x > 1$ $\quad F_X(x) = \int_1^x f_X(u)\,du = 4\int_1^x u^{-5}\,du = 4\left(\frac{1}{-4}\right)(u^{-4})_{u=1}^{x}$

$$= -(x^{-4} - 1) = 1 - x^{-4}$$

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Problem set 3

The tail formula for expectation.

The Weibull distribution. The nonnegative random variable X is said to have the *Weibull distribution* is its cumulative distribution function is of the form

$$F_X(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\tau}}$$

Remark 3.1. In the special case that $\tau = 1$, we get the *exponential distribution*.

Problem 3.1. Let $X \sim Weibull(\theta = 1, \tau = 2)$. What is the expectation of X?

$\rightarrow$: By the tail formula for the expectation,

$$\mathbb{E}[X] = \int_0^{\infty} S_X(x)\,dx = \int_0^{\infty} e^{-x^2}\,dx$$

Looks like the density of a normal dist'n.
For the density of a standard normal dist'n:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

$$x = \frac{u}{\sqrt{2}} \iff u = x\sqrt{2}$$

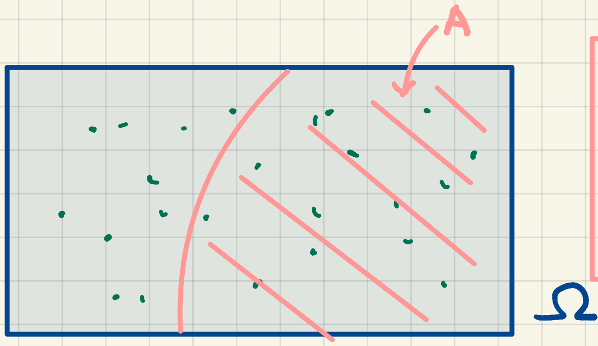
$$dx = \frac{1}{\sqrt{2}}\,du \longleftrightarrow du = \sqrt{2}\,dx$$

$$\int_0^{+\infty} e^{-\frac{u^2}{2}} \left(\frac{1}{\sqrt{2}}\right) du = \frac{1}{\sqrt{2}} \boxed{\int_0^{+\infty} e^{-\frac{u^2}{2}}\,du}$$

$$= \frac{\sqrt{\cancel{2}\pi}}{\sqrt{\cancel{2}}} \underbrace{\boxed{\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{u^2}{2}}\,du}}_{= \frac{1}{2}} = \frac{\sqrt{\pi}}{2} \quad \square$$

Conditioning on an Event.

Let X be a random variable w/ a finite expectation.
Let A be an event such that $\mathbb{P}[A] > 0$.



$$\boxed{\mathbb{E}[X \mid A] := \frac{\mathbb{E}[X \cdot \mathbb{I}_A]}{\mathbb{P}[A]}}$$

Example. X ... severity
d ... deductible

$$\boxed{\mathbb{E}[X - d \mid X > d]}$$

Tail·Value@Risk

By def'n.

$$\boxed{\mathbb{E}\left[X \mid X > VaR_p(X)\right] =: TVaR_p(X)}$$

Example. Let X be a continuous r.v. w/ pdf f_X.

$$TVaR_p(X) = \frac{\mathbb{E}\left[X \cdot \mathbb{I}_{[X > VaR_p(X)]}\right]}{\mathbb{P}[X > VaR_p(X)]} = \frac{\int_{VaR_p(X)}^{+\infty} x f_X(x)\,dx}{p}$$

$$\mathbb{P}[X > VaR_p(X)] = p$$

integration · by · part

$$\boxed{TVaR_p(X) = VaR_p(X) + \frac{1}{p}\int_{VaR_p(X)}^{+\infty} S_X(x)\,dx}$$

Example. [The Exponential Distribution]

$X \sim \text{Exponential}(\text{mean} = \theta)$

$$TVaR_p(X) \underset{\uparrow \text{ by def'n}}{=} \mathbb{E}[X \mid X > VaR_p(X)]$$

constant

$$= \mathbb{E}[(X - VaR_p(X)) + VaR_p(X) \mid X > VaR_p(X)]$$

$$= \underbrace{\mathbb{E}[X - VaR_p(X) \mid X > VaR_p(X)]}_{= \theta} + VaR_p(X)$$

$\leftarrow$ Memoryless Property

$$\boxed{TVaR_p(X) = \theta + VaR_p(X)}$$

$\square$

Strong Law of Large Numbers.

Let $\{X_k, k = 1, 2, \dots\}$ be a sequence of independent, identically distributed r.v.s

Assume the expectation exists and it is finite.
Set

$$\boxed{\mu_X := \mathbb{E}[X_1]} < \infty$$

Then,

$$\boxed{\frac{X_1 + X_2 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{} \underline{\mu_X}}$$

If a function g is such that $g(X_1)$ is well defined, we also have

$$\boxed{\frac{g(X_1) + g(X_2) + \cdots + g(X_n)}{n} \xrightarrow[n\to\infty]{} \underline{\mathbb{E}[g(X_1)]}}$$