M339J: February 27th, 2023.

170. In a certain town the number of common colds an individual will get in a year follows a Poisson distribution that depends on the individual's age and smoking status. The distribution of the population and the mean number of colds are as follows:

	Proportion of population	Mean number of colds
Children	0.30	3 $\lambda_C = 3$
Adult Non-Smokers	0.60	1 $\lambda_{ANS} = 1$
Adult Smokers	0.10	4 $\lambda_{AS} = 4$

Calculate the conditional probability that a person with exactly 3 common colds in a year is an adult smoker.

$\mathbb{P}[AS \mid 3C] = ?$

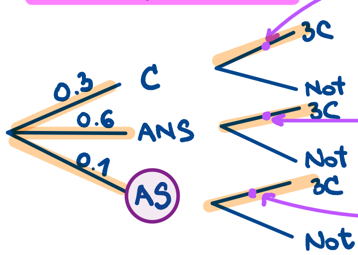
(A) 0.12

(B) 0.16

(C) 0.20

(D) 0.24

(E) 0.28



w/ probab. $e^{-3} \cdot \frac{3^3}{6}$

w/ probab. $e^{-1} \cdot \frac{1^3}{6}$

w/ probab. $e^{-4} \cdot \frac{4^3}{6}$

171. For aggregate losses, S:

(i) The number of losses has a negative binomial distribution with mean 3 and variance 3.6.

(ii) The common distribution of the independent individual loss amounts is uniform from 0 to 20.

Calculate the 95th percentile of the distribution of S as approximated by the normal distribution.

(A) 61

(B) 63

(C) 65

(D) 67

(E) 69

Bayes' Theorem.

$$\mathbb{P}[AS \mid 3C] = \frac{\mathbb{P}[AS \cap 3C]}{\mathbb{P}[C \cap 3C] + \mathbb{P}[ANS \cap 3C] + \mathbb{P}[AS \cap 3C]}$$

$$= \frac{0.1 \cdot e^{-4} \cdot \frac{4^3}{\cancel{6}}}{0.3 \cdot e^{-3} \cdot \frac{3^3}{\cancel{6}} + 0.6 \cdot e^{-1} \cdot \frac{1}{\cancel{6}} + 0.1 \cdot e^{-4} \cdot \frac{4^3}{\cancel{6}}}$$

$$= \underline{0.1581} \quad \square$$

nontypical

$X \sim \text{Poisson}(\lambda=3)$

284. A risk has a loss amount that has a Poisson distribution with mean 3.

An insurance covers the risk with an ordinary deductible of 2. An alternative insurance replaces the deductible with coinsurance α, which is the proportion of the loss paid by the insurance, so that the expected insurance cost remains the same.

Calculate α.

$\longrightarrow$: With an ordinary deductible $d=2$, the expected insurance cost is

$$\mathbb{E}[(X-d)_+] = \underbrace{\mathbb{E}[X]}_{=3} - \boxed{\mathbb{E}[X \wedge 2]}$$

(A) 0.22

(B) 0.27

(C) 0.32

(D) 0.37

(E) 0.42

285. You are the producer for the television show Actuarial Idol. Each year, 1000 actuarial clubs audition for the show. The probability of a club being accepted is 0.20.

The number of members of an accepted club has a distribution with mean 20 and variance 20. Club acceptances and the numbers of club members are mutually independent.

Your annual budget for persons appearing on the show equals 10 times the expected number of persons plus 10 times the standard deviation of the number of persons.

Calculate your annual budget for persons appearing on the show.

(A) 42,600

(B) 44,200

(C) 45,800

(D) 47,400

(E) 49,000

$$X \wedge 2 \sim \begin{cases} 0 & \text{w/ probab.} \\ 1 & \text{w/ probab.} \\ 2 & \text{w/ probab.} \end{cases}$$

$p_X(0) = p_0 = e^{-3}$

$p_X(1) = p_1 = e^{-3} \cdot \frac{3^1}{1!} = 3e^{-3}$

$\mathbb{P}[X \geq 2] = 1 - p_0 - p_1$

$= 1 - 4e^{-3}$

$$\mathbb{E}[X \wedge 2] = 0 \cdot e^{-3} + 1 \cdot (3e^{-3}) + 2 \cdot (1 - 4e^{-3})$$

$$= 2 - 5e^{-3}$$

$$\Rightarrow \mathbb{E}[(X-2)_+] = 3 - (2 - 5e^{-3}) = 1 + 5e^{-3} = 1.25$$

We the second insurance policy,

$$\alpha \cdot \mathbb{E}[X] = 3 \cdot \alpha = 1.25$$

$$\alpha = \frac{1.25}{3} = 0.42$$

Challenge!

130. Bob is a carnival operator of a game in which a player receives a prize worth $W = 2^N$ if the player has N successes, $N = 0, 1, 2, 3,\ldots$ Bob models the probability of success for a player as follows:

(i) N has a Poisson distribution with mean Λ.

(ii) Λ has a uniform distribution on the interval (0, 4).

Calculate $E[W]$.

(A) 5

(B) 7

(C) 9

(D) 11

(E) 13

131. DELETED

132. DELETED

Theorem.

Let $N_1, N_2, \ldots, N_\ell$ be independent, Poisson random variables w/ parameters $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, resp.

Set $\boxed{N := N_1 + N_2 + \cdots + N_\ell}$ ✓

Then:

$$N \sim \text{Poisson}(\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell)$$

Proof.

Focus on the pgf of N.

$$P_N(z) = \mathbb{E}[z^N] = \mathbb{E}[z^{N_1 + N_2 + \cdots + N_\ell}]$$

$$= \mathbb{E}[z^{N_1} \cdot z^{N_2} \cdot \ldots \cdot z^{N_\ell}] \quad \text{(independence)}$$

$$= \mathbb{E}[z^{N_1}] \cdot \mathbb{E}[z^{N_2}] \cdots \mathbb{E}[z^{N_\ell}] =$$

$$= e^{\lambda_1(z-1)} \cdot e^{\lambda_2(z-1)} \cdots e^{\lambda_\ell(z-1)}$$

$$= e^{(\lambda_1 + \lambda_2 + \cdots + \lambda_\ell)(z-1)} = e^{\lambda(z-1)}$$

$\square$

172. A policyholder has probability 0.7 of having no claims, 0.2 of having exactly one claim, and 0.1 of having exactly two claims. Claim amounts are uniformly distributed on the interval [0, 60] and are independent. The insurer covers 100% of each claim.

Calculate the probability that the total benefit paid to the policyholder is 48 or less.

(A) 0.320
(B) 0.400
(C) 0.800
(D) 0.892
(E) 0.924

173. In a given region, the number of tornadoes in a one-week period is modeled by a Poisson distribution with mean 2. The numbers of tornadoes in different weeks are mutually independent.

Calculate the probability that fewer than four tornadoes occur in a three-week period.

(A) 0.13
(B) 0.15
(C) 0.29
(D) 0.43
(E) 0.86

$\rightarrow$: N... # of tornadoes in a three-week period

$\mathbb{P}[N<4] = ?$

$N = N_1 + N_2 + N_3$ w/ $N_i \sim \text{Poisson}(\lambda_i = 2)$ and independent

$\Rightarrow \boxed{N \sim \text{Poisson}(\lambda = 2 \cdot 3 = 6)}$

174. An electronic system contains three cooling components that operate independently. The probability of each component's failure is 0.05. The system will overheat if and only if at least two components fail.

Calculate the probability that the system will overheat.

(A) 0.007
(B) 0.045
(C) 0.098
(D) 0.135
(E) 0.143

$$\mathbb{P}[N<4] = p_0 + p_1 + p_2 + p_3$$

$$= e^{-6} + e^{-6} \cdot \frac{6^1}{1!} + e^{-6} \cdot \frac{6^2}{2!} + e^{-6} \cdot \frac{6^3}{3!}$$

$$= e^{-6} + 6e^{-3} + 18e^{-6} + 36e^{-6}$$

$$= 61e^{-6} = 0.1512$$

Think about:
Say that we have a model for the total count of events coming from several different categories. Say that the total count is Poisson. Assume that you know the proportions of events from different categories.
What is the model for the counts from particular individual categories?